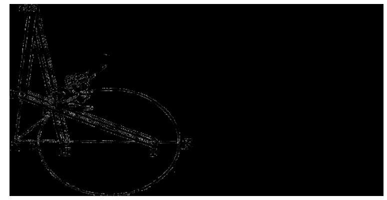
Square Root Calculations



In this investigation we will examine different historical methods of computing square roots. It is important to remember that the ancient Hindus and the medieval Europeans did not use decimals or any other place value system for fractions. The approximations had to be given as ratios of integers. One way to think of a ratio in context is as a scaling factor, i.e. so many of these... to so many of these... This makes particularly good sense if one is working with physical materials as in architecture.

A. An ancient Babylonian approach to computing a square root.

One approach that seems to have been taken by many ancient Babylonian scribes was the following. Suppose that we seek $\sqrt{N}$. First make a guess, which we will call G_1. Next compare G_1 with $\frac{N}{G_1}$. If they are the same you are done; if not, average the two and use that value as your next guess. Call the average G_2. Compare the value of G_2 with $\frac{N}{G_2}$. If they are different, average them to get G_3. Continue until you achieve the level of accuracy that you desire.

Example 1. Find the square root of N=11, to the nearest hundredth. Since 11 is between $3^2 = 9$ and $4^2 = 16$, we'll use 3 as a first guess. (Don't spend too much effort getting a first guess.) Results are shown both as fractions and as decimals; decimals are rounded to thousandths, since we wanted hundredths. For some purposes, fractions are more elucidating; for others, decimals are.

Step	Guess, as fraction	Guess, as decimal, rounded	N/guess, as fraction	N/guess, as decimal, rounded	Difference between guess and N/Guess, as decimal, rounded
1	3	3	$\frac{11}{3} = 3\frac{2}{3}$	3.667	0.667
2	$\frac{10}{3} = 3\frac{1}{3}$	3.333	$\frac{33}{10} = 3\frac{3}{10}$	3.3	0.033
3	$\frac{199}{60} = 3\frac{19}{60}$	3.317	$\frac{660}{199} = 3\frac{63}{199}$	3.317	0.000

So the square root of 11, to the nearest hundredth, is 3.32. This algorithm actually gave the result to the nearest thousandth in the third step.

Try this (1). Use paper and pencil decimal arithmetic (or a counting board or abacus, if you prefer to be more authentic) and the Babylonian method to find the following square roots to the specified level of accuracy.

a) $\sqrt{1681}$ exactly (1681 is a perfect square.) Use 41 as your first guess.

b) $\sqrt{676}$ exactly. Find a first guess mentally.

c) $\sqrt{5}$, to the nearest hundredth.

Was the long division as bad as you thought? Why?

Try this (2). (Detour into number theory; optional.) Use paper and pencil *fraction* arithmetic (not decimals) to find the square root of 2, starting with a first guess of 1. It's easier to use improper fractions. Find a pattern in the numerators and denominators of the guesses; do as many steps as you need to detect a pattern. Can you express this pattern algebraically? Does it adapt to the square roots of other integers?

Try this (3). Experiment with the GeoGebra (spreadsheet) file BabylonSqrt.html. Think about these questions.

a) How many decimal places of accuracy do you get at each step?

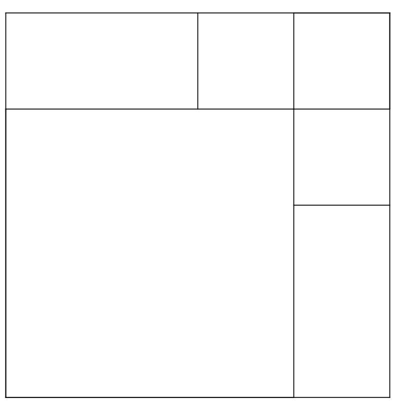
b) How critical is it to get a good first guess? Compute the square root of the same number with various first guesses, and see how many steps it takes to get a specified level of accuracy. Try some really terrible first guesses, like 10,000 for the square root of 2.

c) Find the square roots of some really large (such as 1,428,036) and some really small (such as 0.00000000023456) numbers, and use these results to practice estimating square roots of large and small numbers.

B. An ancient Hindu square root method

The following is an example of a general geometrical method used by the ancient Hindus for finding a series of rational approximations to square roots. It was described by Baudhayana in the Sulbasutra (800-500 BC). The method has the flavor of carpentry or stonemasonry in that it gives the size of a series of trimmings to be made to a square that is too large. However the exactitude of the calculations gives this method a mathematical value that goes beyond its architectural origins.

Starting with a 2 x 2 square with an area of 4 sq. units, we seek to construct a square with an area of 2 sq. units by recursively trimming off excess area.



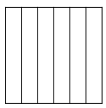
Starting with a 2 x 2 square,
Excess Area = 2 sq. units

Trim off strips of width 1/2 (four pieces)

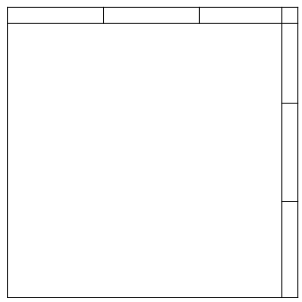
Overlap of 1/2 x 1/2 means that we did not trim off enough.

$$\text{Side of new smaller square } = 2 - \frac{1}{2} = \frac{3}{2} = 1.5$$

Continuing in this fashion, we take the excess area and divide it up into strips which can be laid out to fit exactly along two sides of the square in an attempt to trim off the appropriate amount of excess area. The excess area at this step is a $\frac{1}{2} \times \frac{1}{2}$ square. We now divide this square into 6 strips (each $\frac{1}{2} \times \frac{1}{12}$) and trim off the square.



Excess Area divided into six equal strips



1.5 x 1.5 square

Excess Area = 1/2 x 1/2 = 1/4 sq. units

Trim off strips of width 1/12 (six pieces)

Over lap of 1/12 x 1/12 means that again we did not trim off enough.

$$\text{Side of new smaller square } = 2 - \frac{1}{2} - \frac{1}{2 \cdot 6} = 2 - \frac{1}{2} - \frac{1}{12} = \frac{17}{12} = 1 + \frac{5}{12} \approx 1.41666667$$

Continuing in this fashion, we again take the excess area and divide it up into strips that can be laid out to fit exactly along two sides of the square in an attempt to trim off the appropriate amount of excess area. The new excess area is a $\frac{1}{12} \times \frac{1}{12}$ square. We now divide this square into 34 strips (each of size $\frac{1}{12} \times \frac{1}{408}$). Fitting 17 of these strips on each side, we again trim off the square.

$$\begin{aligned} \text{Side of new smaller square } &= 2 - \frac{1}{2} - \frac{1}{2 \cdot 6} - \frac{1}{2 \cdot 6 \cdot 34} \\ &= 2 - \frac{1}{2} - \frac{1}{12} - \frac{1}{408} \\ &= \frac{577}{408} = 1 + \frac{169}{408} \approx 1.41215686 \end{aligned}$$

Continuing in this fashion, we again take the excess area and divide it up into strips that can be laid out to fit exactly along two sides of the square in an attempt to trim off the appropriate amount of excess area. The new excess area is a $\frac{1}{408} \times \frac{1}{408}$ square. We now divide this square into 1154 strips (each of size $\frac{1}{408} \times \frac{1}{470832}$) and again trim off the square.

$$\text{Side of new smaller square } = 2 - \frac{1}{2} - \frac{1}{2\cdot 6} - \frac{1}{2\cdot 6\cdot 34} - \frac{1}{2\cdot 6\cdot 34\cdot 1154}$$

$$= 2 - \frac{1}{2} - \frac{1}{12} - \frac{1}{408} - \frac{1}{470832}$$

$$= \frac{665857}{470832} = 1 + \frac{195025}{470832} \approx 1.414213562$$

Try this (4). Use the Hindu method to find the square root of another integer, such as 3 or 5.

Try this (5). Compare with the steps for the square root of the same number, using Try this (2).

Try this (6). Experiment with various numbers in HinduSqrt.html to *see* the size of the pieces that are being removed at each step. This demonstration shows 3 steps of trimming. Zoom in on the upper right corner to see the overlap.

C. A square root method used by 10th century European architects.

Recently an architecture student at UCLA, Sukuru Yuksel, investigated how tenth century European architects approximated square roots. He came upon the following procedure.

Consider the geometric sequence 1, 2, 4, 8, 16, 32, . . . with a constant ratio of 2. Next consider how one might insert geometric means into the sequence. That is to say how might one refine or interpolate the sequence in such a way that it remains a geometric sequence (i.e. consecutive terms have a constant ratio). What is wanted is the sequence:

$$1,\ \sqrt{2},\ 2,\ 2\sqrt{2},\ 4,\ 4\sqrt{2},\ 8,\ 8\sqrt{2},\ 16,\ 16\sqrt{2},\ 32,\ \ldots$$

Now one starts by making a guess at $\sqrt{2}$. For example let us start with the crude guess $\sqrt{2} \approx 1$. The sequence then becomes: 1, 1, 2, 2, 4, 4, 8, 8, 16, 16, 32, . . . Starting with this sequence, one then forms a new sequence in which each new term is the sum of two consecutive terms in the previous sequence (in the same way that one generates Pascal's triangle). By doing this each new sequence comes closer and closer to having a common ratio of $\sqrt{2}$. For example:

1 1 2 2 4 4 8 8 16 16 32 32
2 3 4 6 8 12 16 24 32 48 64
5 7 10 14 20 28 40 56 80 112
12 17 24 34 48 68 96 136 192
29 41 58 82 116 164 232 328
70 99 140 198 280 396 560
169 239 338 478 676 956
408 577 816 1154 1632
985 1393 1970 2786
2378 3363 4756
5741 8119

Don't do the final addition. The square root of 2 is approximately the ratio of the final two numbers: $\frac{8119}{5714} = 1.414213552\ldots$. A modern calculator gives $\sqrt{2} \approx 1.414213562$. That is to say, the last ratio in the table above is an approximation to $\sqrt{2}$ that is accurate to seven decimal places.

Note that this algorithm could be performed easily with Roman numerals.

See the GeoGebra file EurSqrt.html.

Try this (7).

a) If you haven't already done this, use each of the three methods to approximate the square root of the same number, such as $\sqrt{2}$ or $\sqrt{5}$. Compare and contrast the methods. Consider the issues of format, simplicity, accuracy, and efficiency. Comment on any patterns that you notice in any of the approximation schemes.

b) Experiment with using different first guesses in three different methods. What is the effect of using a different starting point?

c) Try to come up with some reasoning that would justify each of the three methods.

d) Is it possible to modify any of the three processes so as to compute rational approximations to a cube root? How might you go about doing this?

Try this (8). (For further practice) Construct your own spreadsheets for each of the three methods.