Appendices to The Creation of Continuous Exponents by David Dennis

Appendix 1: The Binomial Series of Isaac Newton

In 1661, the nineteen-year-old Isaac Newton read the *Arithmetica Infinitorum* and was much impressed. In 1664 and 1665 he made a series of annotations from Wallis which extended the concepts of interpolation and extrapolation. It was here that Newton first developed his binomial expansions for negative and fractional exponents and these early papers of Newton are the primary source for our next discussion (Newton, 1967a, Vol. 1, p 89-142).

Newton made a series of extensions of the ideas in Wallis. He extended the tables of areas to the left to include negative powers and found new patterns upon which to base interpolations. Perhaps his most significant deviation from Wallis was that Newton abandoned the use of ratios of areas and instead sought direct expressions which would calculate the area under a portion of a curve from the value of the abscissa. Using what he knew from Wallis he could write down area expressions for the integer powers. Referring back to figure 1, we have:

$$\frac{\text{Area under } x^n}{\text{Area of containing rectangle}} = \frac{1}{n+1} \,, \qquad \text{and}$$

$$\text{Area of the containing rectangle} = x \cdot x^n = x^{n+1} \,, \qquad \text{hence,}$$

$$\text{the Area under the curve } x^n = \frac{x^{n+1}}{n+1} \,.$$

Using this form combined with binomial expansions, Newton wrote down progressions of expressions which calculated the area under curves in particular families. For example, he considered the positive and negative integer powers of 1+x, i.e. the series of curves :

$$\ldots\ldots \quad y=\frac{1}{1+x}\,, \quad y=1, \quad y=1+x\,, \quad y=(1+x)^2, \quad y=(1+x)^3, \quad y=(1+x)^4, \ldots\ldots$$

He was particularly interested in the hyperbola and wanted to find its area expression by interpolation after having failed to obtain its area by purely geometric considerations (Newton, 1967a, Vol. 1, p 94).

Newton drew the following graph of several members of this family of curves (figure 4). Appearing in the graph are a hyperbola, a constant, a line, and a parabola, i.e. the first four curves in the progression..

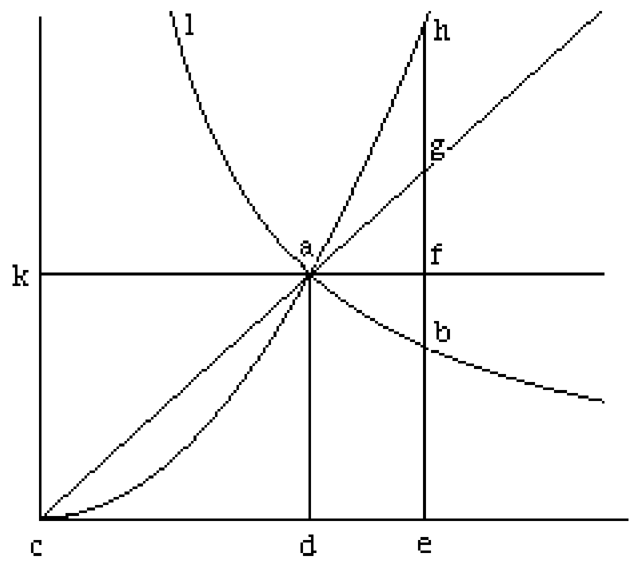

Fig. 4

Letting ck=cd=1 and de = x , the ordinates here are: eb=$\frac{1}{1+x}$, ef=1, eg=1+x, eh=$(1+x)^2$. He then wrote down a series of expressions which calculate the areas under the curves over the segment de=x as:

Area(afed)= x, Area(aged)= $x + \frac{x^2}{2}$, Area(ahed)= $x + \frac{2x^2}{2} + \frac{x^3}{3}$.

The third one is obtained by first expanding $(1+x)^2$, as $1+2x+x^2$ (see appendix 3). Although the higher power curves did not appear in the graph Newton went on to write down more area expressions for curves in this family. For the positive integer powers 3, 4, and 5 of (1+x) he obtained the following area expressions by first expanding and then finding the area term by term.

(third power) $x + \frac{3x^2}{2} + \frac{3x^3}{3} + \frac{x^4}{4}$, (fourth power) $x + \frac{4x^2}{2} + \frac{6x^3}{3} + \frac{4x^4}{4} + \frac{x^5}{5}$,

(fifth power) $x + \frac{5x^2}{2} + \frac{10x^3}{3} + \frac{10x^4}{4} + \frac{5x^5}{5} + \frac{x^6}{6}$.

At this point Newton wanted to find a pattern which would allow him to extend his calculations to include the areas under the negative powers of (1+x). He noticed that the denominators form an arithmetic sequence while the numerators follow the binomial patterns. This binomial pattern in the numerators is not so surprising given that they came from expansions. He then made the following table of the area expressions for $(1+x)^p$ (see table 4) , where each column represents the numbers in the numerators of the area function. The question then becomes: how can one fill in the missing entries? He began by assuming that the top row remains constant at the value 1.

Table 4

p / term	-4	-3	-2	-1	0	1	2	3	4	5	6
$\frac{x}{1}$	1	1	1	1	1	1	1	1	1	1	1
$\frac{x^2}{2}$				**?**	0	1	2	3	4	5	6
$\frac{x^3}{3}$					0	0	1	3	6	10	15
$\frac{x^4}{4}$					0	0	0	1	4	10	20
$\frac{x^5}{5}$					0	0	0	0	1	5	15
$\frac{x^6}{6}$					0	0	0	0	0	1	6
$\frac{x^7}{7}$					0	0	0	0	0	0	1

This binomial table is different from Wallis' table in that the rows are all nudged successively to the right so that the diagonals of the Wallis table become the columns of Newton's table. The binomial pattern of formation is now such that each entry is the sum of the entry to the left of it and the one above that one. Using this rule backwards as a difference we find, for example, that the **?** must be equal to -1. Each new diagonal to the left is the sequence of differences of the previous diagonal. This was Newton's first use of difference tables. Continuing on in a similar manner Newton filled in the table of coefficients for the area expressions under the curves $(1+x)^p$ as follows:

Table 5

p / term	-4	-3	-2	-1	0	1	2	3	4	5	6
$\frac{x}{1}$	1	1	1	1	1	1	1	1	1	1	1
$\frac{x^2}{2}$	-4	-3	-2	-1	0	1	2	3	4	5	6
$\frac{x^3}{3}$	10	6	3	1	0	0	1	3	6	10	15
$\frac{x^4}{4}$	-20	-10	-4	-1	0	0	0	1	4	10	20
$\frac{x^5}{5}$	35	15	5	1	0	0	0	0	1	5	15
$\frac{x^6}{6}$	-56	-21	-6	-1	0	0	0	0	0	1	6
$\frac{x^7}{7}$	84	28	7	1	0	0	0	0	0	0	1

At this point Newton could write down the area under the hyperbola: $y=\frac{1}{1+x}$, (i.e. what we now call the natural logarithm of 1+x)(see figure 4) as:

$$\text{(6)} \qquad \text{Area(abed)}= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} \ldots$$

He then made several detailed calculations using the first 25 terms of this series to compute hyperbolic areas to more than 50 decimal places. Newton later became aware that this function displayed logarithmic properties and could be used to create a table of common logarithms (Edwards, 1979, p. 160).

Newton repeatedly returned to the table of characteristic ratios made by Wallis (table 3). As discussed previously, Newton abandoned Wallis' use of area ratios and set out to make a table of coefficients for a sequence of explicit expressions for calculating areas. He used the same set of curves whose characteristic ratios Wallis had tabulated in the row q=1/2, but Newton let r=1 (in the circle case r is the radius). Hence he considered the areas (over the segment de=x) under the following sequence of curves (see figure 5):

$$\ldots, \quad y=1, \quad y=\sqrt{1-x^2}\,, \quad y=1-x^2, \quad y=(1-x^2)\sqrt{1-x^2}\,, \quad y=(1-x^2)^2, \ldots$$

These are the powers of $(1-x^2)$ at intervals of 1/2. In this early manuscript Newton did not write fractions directly as exponents, but when he later announced the results of his researches in a series of letters he did, thus $(1-x^2)\sqrt{1-x^2}$ would become $(1-x^2)^{3/2}$. Several times Newton drew graphs of these curves inside the unit square (see figure 5).

(Newton's original manuscript containing one of these graphs is on the cover of this paper.)

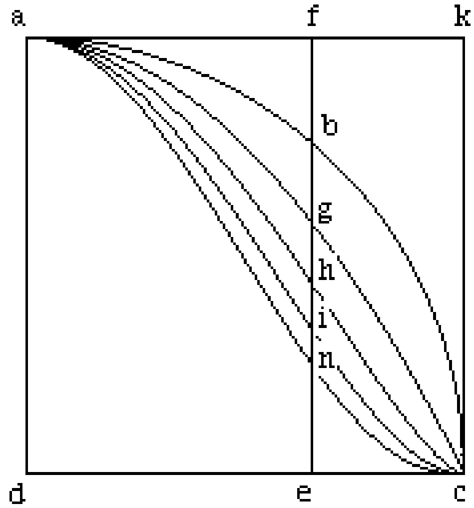

Fig. 5

He let ad=dc=1 and de=x ; ef, eb, eg, eh, ei, en, . . . are then the ordinates of his series of curves respectively. Note that the curve abc is a circle, and agc is a parabola.

For the <u>integer</u> powers of $(1-x^2)$, Newton could write down the areas in his graph (figure 5) as:

Area(afed) = x, Area(aged) = $x - \frac{1}{3}x^3$, Area(aied) = $x - \frac{2}{3}x^3 + \frac{1}{5}x^5$

As before, these are obtained by first expanding the binomials and then writing down the area expressions term by term. Once again he applied the characteristic ratios of Wallis to each separate term in the expansion (see appendix 3). Although the higher powers no longer appeared in his graph Newton continued this sequence of area expressions for $(1-x^2)^p$ as follows:

(p=3) $x - \frac{3}{3}x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7$, (p=4) $x - \frac{4}{3}x^3 + \frac{6}{5}x^5 - \frac{4}{7}x^7 + \frac{1}{9}x^9$,

(p=5) $x - \frac{5}{3}x^3 + \frac{10}{5}x^5 - \frac{10}{7}x^7 + \frac{5}{9}x^9 - \frac{1}{11}x^{11}$, etc.

Once again he saw that the denominators formed an arithmetic sequence and that the numerators followed a binomial pattern. As before Newton made a table of these results including an extension into the negative powers. Table 6 is a table of coefficients of the expressions which compute the area under the curves $y = (1-x^2)^p$.

Table 6

p / term	-1	$\frac{-1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
$\frac{x}{1}$	1		1		1		1		1		1
$\frac{-x^3}{3}$	-1		0		1		2		3		4
$\frac{x^5}{5}$	1		0		0		1		3		6
$\frac{-x^7}{7}$	-1		0		0		0		1		4
$\frac{x^9}{9}$	1		0		0		0		0		1
$\frac{-x^{11}}{11}$	-1		0		0		0		0		0
$\frac{x^{13}}{13}$	1		0		0		0		0		0

It now remained to find a way to interpolate the missing entries for the fractional powers. In this table each entry is the sum of the entry two spaces to the left and the entry directly above that one. The entries above the diagonal of 1's had already been interpolated by Wallis in table 3, and from these one could complete the table by differences as in table 5. One could also have used the polynomials that appeared in the margins of Wallis' table 3 to fill in this table. Newton, however, devised his own system of interpolation which he could check against these others. Instead of forming polynomial expressions for the interpolation of each row Newton used the known entries to generate a system of linear equations whose solution would determine the missing entries.

He first noted that integer binomial tables obey the following additive pattern of formation (table 7).

Table 7

a	a	a	a	a
b	a+b	2a+b	3a+b	4a+b
c	b+c	a+2b+c	3a+3b+c	6a+4b+c
d	c+d	b+2c+d	a+3b+3c+d	4a+6b+4c+d
e	d+e	c+2d+e	b+3c+3d+e	a+4b+4c+4d+e

This pattern is formed by starting with a constant sequence (a,a,a,...) and an arbitrary left hand column (a,b,c,d,...); and then forming each entry as the sum of the one to the left and the one above that. This, as it stands, would not work for the completion of the fractional interpolated tables, because the entries in the top row must all be 1 in all the interpolated tables (i.e. a=1), but this would force the increment of the second row also to be one. To get around this difficulty, Newton rewrote this pattern so as to unlink the rows of table 7. That is to say, he preserved the pattern within each individual row but he changed the names of the variables so that each variable appeared in only one row. As you move down the rows each new row can be described using successively one more variable. Changing the names of variables so that each row is independent of the others, the pattern now becomes table 8.

Table 8

a	a	a	a	a
b	c+b	2c+b	3c+b	4c+b
d	e+d	f+2e+d	3f+3e+d	6f+4e+d
g	h+g	i+2g+h	k+3i+3h+g	4k+6i+4h+g
l	m+l	n+2m+l	p+3n+3m+l	q+4p+6n+6m+l

Using table 8, if any entry in the first row is known the whole row is known. If any two entries in the second row are known then one can solve for b and c and fill in the entire row. If any three entries in the third row are known one can solve for d, e and f and fill in the entire row. Thus with a sufficient number of known values in a given row one could solve a system of linear equations for all the variables in that row. Newton solved sets of linear equations to find these values and that allowed him to fill in the interpolated table. This method allowed him not only to interpolate the binomial table at increments of 1/2, but at any increment, for example, thirds.

He then completed table 6. Let us complete the third row, for example, using the known values 0, ?, 0, ?, 1, ? We obtain d=0, f+2e+d=0, and 6f+4e+d=1. Thus d=0, $f = \frac{1}{4}$, and $e = \frac{-1}{8}$. We can now complete the entire row using these values, but it should be noted here that although we used three equations to find d, e, and f there are actually an infinite number of equations involving these three variables and one might ask if this set of equations is consistent. They are, but Newton did not address this issue. He is satisfied because the values he finds agree with Wallis and with the additive pattern of table formation. With the completion of table 6, Newton will also obtain a new way to calculate π which will validate his method in a geometric representation. Table 6 now becomes:

Table 9

p / term	-1	$\frac{-1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
$\frac{x}{1}$	1	1	1	1	1	1	1	1	1	1	1
$\frac{-x^3}{3}$	-1	$\frac{-1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
$\frac{x^5}{5}$	1	$\frac{3}{8}$	0	$\frac{-1}{8}$	0	$\frac{3}{8}$	1	$\frac{15}{8}$	3	$\frac{35}{8}$	6
$\frac{-x^7}{7}$	-1	$\frac{-5}{16}$	0	$\frac{3}{48}$	0	$\frac{-1}{16}$	0	$\frac{5}{16}$	1	$\frac{35}{16}$	4
$\frac{x^9}{9}$	1	$\frac{35}{128}$	0	$\frac{-15}{384}$	0	$\frac{3}{128}$	0	$\frac{-5}{128}$	0	$\frac{35}{128}$	1
$\frac{-x^{11}}{11}$	-1	$\frac{-63}{256}$	0	$\frac{105}{3840}$	0	$\frac{-3}{256}$	0	$\frac{3}{256}$	0	$\frac{-7}{256}$	0
$\frac{x^{13}}{13}$	1	$\frac{231}{1024}$	0	$\frac{-945}{46080}$	0	$\frac{7}{1024}$	0	$\frac{-5}{1024}$	0	$\frac{7}{1024}$	0

The column p=1/2 gives an infinite series which calculates the area under any portion of a circle (see figure 5). That is to say that Area(abed) is given by (7), where de=x.

$$(7) \qquad x - \frac{1}{2}\cdot\frac{x^3}{3} - \frac{1}{8}\cdot\frac{x^5}{5} - \frac{3}{48}\cdot\frac{x^7}{7} - \frac{15}{384}\cdot\frac{x^9}{9} - \frac{105}{3840}\cdot\frac{x^{11}}{11} - \cdots$$

Letting x=1 in this series calculates the area of one quarter of the circle and thus yields a new calculation of π:

$$(8) \qquad \frac{\pi}{4} = 1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \frac{5}{1152} - \frac{7}{2816} - \cdots$$

Checking that this series does agree with the value of π obtained from geometrical arguments like those of Archimedes, as well as the infinite product of Wallis; provided Newton with a validation of this interpolation in alternate representations.

Newton later became aware that the interpolation procedure based on the patterns of table 8 was equivalent to the assumption that rows of this table could be interpolated using polynomial equations of increasing degree. That is to say, the first row is constant, the second row is linear, the third row is quadratic, and so on. This is consistent with the method used by Wallis, and would suggest to Newton a general procedure for the interpolation of data which we will describe in the next section.

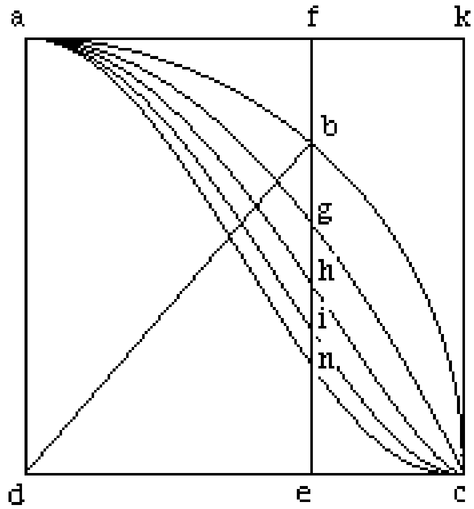

Fig. 6

Newton also pointed out that this series allowed him to compute the arcsin(x). By adding a line from d to b in figure 5 (see figure 6), and subtracting the area of the triangle(dbe) from the Area(abed), one obtains the area of the circular sector(abd). Since this is the circle of radius one, twice the area of the sector(abd) equals the arclength(ab) (when r=1, area=π, and circum.=2π). The triangle(dbe) to be subtracted from the series has area equal to

$$\frac{1}{2}\ x\sqrt{1-x^2}\,.$$

Satisfied with his interpolation methods Newton began searching for a pattern in the columns of his table which would allow him to continue each series without having to repeat his tedious interpolation procedure row by row. Note that some of the fractions in table 9 are not reduced. In earlier tabulations Newton did reduce the fractions but he soon became aware that this would only obscure any possible patterns in their formations. Following the example set by Wallis he sought a pattern of continued multiplication of arithmetic sequences. Since the circle was so important to him he studied the p=1/2 column first. Factoring the numbers in these fractions he found that they could be produced by continued multiplication as:

$$\text{(9)}\qquad \frac{1}{1}\cdot\frac{1}{2}\cdot\frac{-1}{4}\cdot\frac{-3}{6}\cdot\frac{-5}{8}\cdot\frac{-7}{10}\cdot\frac{-9}{12}\cdot\frac{-11}{14}\ \cdot\cdot\cdot\cdot$$

Similarly the entries in the p=3/2 column could be produced by continued multiplication as:

$$\text{(10)}\qquad \frac{1}{1}\cdot\frac{3}{2}\cdot\frac{1}{4}\cdot\frac{-1}{6}\cdot\frac{-3}{8}\cdot\frac{-5}{10}\cdot\frac{-7}{12}\cdot\frac{-9}{14}\ \cdot\cdot\cdot\cdot$$

In order to further investigate these patterns, Newton carried out an interpolation of the binomial table at intervals of 1/3. Using the patterns from table 8 and solving the systems of equations for the variables in each row he produced the following interpolated table 10. Note that at this point he does not write down the terms in the expansions for which these numbers are coefficients. Newton never mentions an explicit context of area calculations for which table 10 was intended. At this point he is working solely within a table representation in order to find an explicit formula for the fractional binomial numbers whose patterns began revealing themselves in (9) and (10). After another long round of solving systems of linear equations, Newton arrived at:

Table 10

0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2	$\frac{7}{3}$	$\frac{8}{3}$	3	$\frac{10}{3}$
1	1	1	1	1	1	1	1	1	1	1
0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2	$\frac{7}{3}$	$\frac{8}{3}$	3	$\frac{10}{3}$
0	$\frac{-1}{9}$	$\frac{-1}{9}$	0	$\frac{2}{9}$	$\frac{5}{9}$	1	$\frac{14}{9}$	$\frac{20}{9}$	3	$\frac{35}{9}$
0	$\frac{5}{81}$	$\frac{4}{81}$	0	$\frac{-4}{81}$	$\frac{-5}{81}$	0	$\frac{14}{81}$	$\frac{40}{81}$	1	$\frac{140}{81}$
0	$\frac{-10}{243}$	$\frac{-7}{243}$	0	$\frac{5}{243}$	$\frac{5}{243}$	0	$\frac{-7}{243}$	$\frac{-10}{243}$	0	$\frac{25}{243}$
0	$\frac{22}{729}$	$\frac{14}{729}$	0	$\frac{-8}{729}$	$\frac{-7}{729}$	0	$\frac{7}{729}$	$\frac{8}{729}$	0	$\frac{-14}{729}$

Searching, as before, for a pattern of repeated multiplication of arithmetic sequences that would generate the columns of this table, Newton discerned the following pattern for the column p=1/3.

(11) $$\frac{1}{1} \cdot \frac{1}{3} \cdot \frac{-2}{6} \cdot \frac{-5}{9} \cdot \frac{-8}{12} \cdot \frac{-11}{15} \cdot \frac{-14}{18} \cdot \frac{-17}{21} \quad . \; . \; . \; .$$

Here the sequence of numerators and denominators both change by increments of 3 (ignoring the first term), the former going down while the later go up. In (9) and (10) the same thing happened but by increments of 2. At this point Newton wrote down an explicit formula for the binomial numbers in an arbitrary column $p = \frac{x}{y}$.

(12) $$\frac{1}{1} \cdot \frac{x}{y} \cdot \frac{x-y}{2y} \cdot \frac{x-2y}{3y} \cdot \frac{x-3y}{4y} \cdot \frac{x-4y}{5y} \cdot \frac{x-5y}{6y} \quad . \; . \; . \; .$$

This formula makes perfect sense given the form of the examples from which it was constructed. If the y's in the denominators are taken into the numerators it becomes the formula for the binomial coefficient that is familiar to a modern reader. Dropping the first term and letting n=x/y, as Newton would do in his later letters, (12) becomes:

(13) $$\frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5} \cdot \frac{n-5}{6} \quad . \; . \; . \; .$$

The binomial series became the engine which generated a wealth of examples from which Newton would later build his version of calculus. Once he had written down (12) and later (13), he began a long series of experiments and checks to convince himself of its validity. For example, by using synthetic division one can write:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \ldots.$$

By computing the area of this series term by term one could then arrive at the series for hyperbolic areas (6). It is important to note here that this is not how Newton first constructed (6) (several history books give that impression as does one of Newton's own accounts). He noticed this later and saw it as an important algebraic confirmation of the validity of his table construction. Thus his extension of the binomial table could be checked against both geometric areas, and algebraic generalizations of arithmetic.

His original interpolations were designed to calculate areas under families of curves but Newton soon saw that by changing the terms to which the coefficients were applied he could use these numbers to calculate the points on the curve as well. This was particularly useful for root extractions. He simply had to replace the area terms $\frac{x^{n+1}}{n+1}$ with the original terms x^n from which they came. The coefficients in the tables remain the same. For example, the p=1/2 column of table 9 can be used to calculate square roots as:

$$(14) \qquad (1-x^2)^{1/2} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 \ldots\ldots$$

$$(15) \qquad (1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 \ldots\ldots$$

The p=1/3 column of table 10 can be used to calculate cube roots as:

$$(16) \qquad (1-x^2)^{1/3} = 1 - \frac{1}{3}x^2 - \frac{1}{9}x^4 - \frac{5}{81}x^6 - \frac{10}{243}x^8 \ldots\ldots$$

$$(17) \qquad (1+x)^{1/3} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 \ldots\ldots$$

These series appear in this form in the letters (1676) to Oldenburg in which Newton explained his binomial series at the request of Leibniz (Callinger, 1982; Struik, 1986). He checked the consistency of these series in many ways. Various geometric methods for finding square roots were known against which (14) and (15) could be checked. The series (15) can be multiplied by itself term by term to arrive at 1+x , all other terms canceling out. Newton never gave anything like a formal proof of the validity of these generalized binomial expansions. His approach was always empirical. He tried them in various contexts and they worked. If certain values were needed for which a particular series diverged, he just rewrote it in another form until he found one that converged. Questions concerning convergence were treated empirically for over a century after Newton. As we shall see in the next discussion, Newton did not distinguish between the interpolation of scientific data and the continuous calculation of mathematically defined curves.

Appendix 2: Newton and Empirical Interpolation

In the second of the two letters to Oldenburg of 1676, Newton remarks, "When simple series are not obtainable with sufficient ease I have another method, not yet published, by which the problem is easily dealt with. It is based upon a convenient, ready, and general solution of the problem. To describe a geometrical curve which shall pass through any given points. . . Although the problem may seem to be intractable at first sight it is never the less quite the contrary. Perhaps indeed it is one of the prettiest problems that I can ever hope to solve." (Fraser, 1927, p.45). By the term "geometrical curve" here Newton means a curve with a polynomial equation. He did eventually publish (1710) his method for finding a polynomial equation that would pass through any given finite set of points in his *Methodus Differentialis* (Newton, 1967b, Vol. 2).

We shall briefly describe one section of this work and show its connection to the binomial series. We seek to show how important finite difference tables were in Newton's work, and to emphasize the empirical groundwork that formed the basis of much of his thinking. As the quote above makes clear, finding say a fourth degree polynomial that passed through a given set of five data points was for Newton the same problem as generating the first five terms of an infinite series. Theoretical expansions and fitting a curve to data were in his mind the same problem. For more details see the excellent article by Fraser (1927).

Newton first presented his method by working out in detail the example of finding a fourth degree polynomial equation that passes through an arbitrary set of five points. He first calculated what he called "divided differences." We shall employ Fraser's notation and make a table to show how these differences are calculated. Let Δ' denote the first divided difference which is the same as the average rate of change between two points. Δ'_2 , Δ'_3, Δ'_4 , shall denote the second, third, and fourth divided differences. Each of these is calculated as the difference in the previous one divided by the overall difference in the x-values which were involved in its formation. Three data points are needed to construct one value of Δ'_2, four are needed for each value of Δ'_3, etc. See table 11. Note that Newton calculated his differences in the opposite order from most modern conventions, but since he is dividing by the x differences his signs will come out the same.

Table 11

x	y	Δ'	Δ'_2	Δ'_3	Δ'_4
p	α				
		$\frac{\alpha-\beta}{p-q}$			
q	β		$\frac{\Delta'(p,q)-\Delta'(q,r)}{p-r}$		
		$\frac{\beta-\gamma}{q-r}$		$\frac{\Delta'_2(p,q,r)-\Delta'_2(q,r,s)}{p-s}$	
r	γ		$\frac{\Delta'(q,r)-\Delta'(r,s)}{q-s}$		$\frac{\Delta'_3(p,q,r,s)-\Delta'_3(q,r,s,t)}{p-t}$
		$\frac{\gamma-\delta}{r-s}$		$\frac{\Delta'_2(q,r,s)-\Delta'_2(r,s,t)}{q-t}$	
s	δ		$\frac{\Delta'(r,s)-\Delta'(s,t)}{r-t}$		
		$\frac{\delta-\varepsilon}{s-t}$			
t	ε				

Given five points only one value of Δ'_4 can be calculated. Hence we seek a function which has that constant value as its fourth divided difference everywhere. Such a function must be a fourth degree polynomial. We then must find its coefficients. Here is how Newton described that procedure. Suppose that $y=a+bx+cx^2+dx^3+ex^4$. When we form first differences, 'a' drops out. When we divide those differences by the changes in x, the denominator must be a factor of the numerator and therefore it can be canceled. This happens at each step, i.e. we lose one more of the polynomial coefficients and the divisors are always factors. Newton wrote the divided differences in his table in polynomial form as follows:

(18)
$$\Delta'(p,q) = b+c(p+q)+d(p^2+pq+q^2)+e(p^3+p^2q+pq^2+q^3)$$
$$\Delta'(q,r) = b+c(q+r)+d(q^2+qr+r^2)+e(q^3+q^2r+qr^2+r^3)$$
$$\Delta'(r,s) = b+c(r+s)+d(r^2+rs+s^2)+e(r^3+r^2s+rs^2+r^3)$$
$$\Delta'(s,t) = b+c(s+t)+d(s^2+st+t^2)+e(s^3+s^2t+st^2+t^3)$$

$$\Delta'_2(p,q,r) = c+d(p+q+r)+e(p^2+pq+q^2+pr+qr+r^2)$$
$$\Delta'_2(q,r,s) = c+d(q+r+s)+e(q^2+qr+r^2+qs+rs+s^2)$$
$$\Delta'_2(r,s,t) = c+d(r+s+t)+e(r^2+rs+s^2+rt+st+t^2)$$

$$\Delta'_3(p,q,r,s) = d+e(p+q+r+s)$$
$$\Delta'_3(q,r,s,t) = d+e(q+r+s+t)$$

$$\Delta'_4(p,q,r,s,t) = e$$

Here we see that the fourth divided difference is the coefficient of the fourth degree term. Knowing 'e' we can now go back to either of Δ'_3 equations and solve for 'd'. Knowing 'e' and 'd' we can go back to any one of the Δ'_2 equations and solve for 'c'. Continuing in this way we can completely determine the desired polynomial. As in his previous table interpolations, Newton proceeded by solving a set of linear equations. Note that once the table of divided differences is made, only one value from each column is needed in order to construct the polynomial. Newton discussed a variety of strategies for using either the lead differences or a set of central differences, depending on the nature of the data involved (Newton, 1967b; Fraser, 1927).

We shall make only one more observation here. Consider the special case were the values of x are evenly spaced at unit intervals beginning with 0. In this case all of the denominators in the Δ' column are 1, and all of the denominators in the Δ'_2 column are 2, and so on. In this case the each divided difference equals the ordinary finite difference divided by n factorial, i.e. $\Delta'_n = \frac{\Delta_n}{n!}$. In this case his fitted polynomial equation will come out as:

(19)
$$y = \alpha + x\,\Delta_1 + \frac{x(x-1)\Delta_2}{2!} + \frac{x(x-1)(x-2)\Delta_3}{3!} + \frac{x(x-1)(x-2)(x-3)\Delta_4}{4!} \quad .....$$

Newton did not write his polynomials in exactly this algebraic form (19), but instead described in detail procedures for how to work from a table of differences. His procedures do imply this form (Fraser, 1927). Looking at (19) one can see both the general form of the binomial coefficients and the form of the Taylor series. Taylor took his inspiration from Newton and wrote his derivations based upon difference tables in 1715 (Callinger, 1982, p.419).

Appendix 3 Euler and the Exponential Base 'e'

In the next generation after Newton, Euler made extensive use of Newton's generalized binomial expansions greatly extending their range and utility. Newton used tables to construct infinite series, but once the method of formation of this series had been made clear Euler began using the series to construct tables. Euler conducted a lengthy series of investigations concerning the questions of which form of binomial expansions are most efficient for the construction of particular tables. From these investigations comes the modern notion of function and most of its attendant notation.

Euler's notations are familiar to us because during the eighteenth century he wrote a series of extremely influential textbooks which greatly standardized mathematical notation. Although the form and notation of Euler has been retained in our modern curriculum, much of the content and spirit of investigation has been lost. Euler believed strongly in empirical methods and this spirit pervades his famous precalculus text of 1748, the *Introductio in Analysin Infinitorum* (Euler, 1988). He felt strongly that the expansion of functions in infinte series is one of the basic tools of <u>precalculus</u>.

Binomial expansions expose many of the most important properties of functions as well as the connections between different functions. Euler was the first person to calculate the number 'e' and show exactly how the hyperbolic area function is a logarithm. This is accomplished entirely using Newton's binomial expansions. Following this Euler extends the binomial series to complex numbers and expands the trigonometric functions. By looking at these series (what we now call the Taylor series for e^x, sin(x), and cos(x)) Euler discusses the connections which allow him to see these function as one family. The main content of Euler's work which is lost in our modern curriculum, is that by using empirical methods and binomial expansions all of these topics can be investigated at an elementary precalculus level.

Let us look at Euler's treatment of exponential functions. In Chapter VI of the *Introductio,* he presents the usual population and compound interest problems. However, he goes on in Chapter VII to derive several series for computing these functions. Consider the function a^x for a>1. Since $a^0=1$, Euler lets $a^w=1+kw$, where "w is an infinitely small number." Here he is approximating a^x with a linear function on a small interval. k is the slope of the curve a^x at the point (0,1). The value of the constant k depends on the base a. (For example, if a=10, then k=2.30258.....) Now Euler expands $a^{wj} = (1+kw)^j$, using the binomial theorem just as Newton would.

$$(20)\quad a^{wj}= (1+kw)^j = 1+ \frac{j}{1}\ kw + \frac{j(j-1)}{1\cdot 2}\ k^2w^2 + \frac{j(j-1)(j-2)}{1\cdot 2\cdot 3}\ k^3w^3 + \ \ldots$$

Next he makes the substitution x=wj, or $j = \frac{x}{w}$, or $w = \frac{x}{j}$, noting here that since w is "infinitely small" we are now supposing that j is "infinitely large". (20) now becomes:

$$(21)\quad a^x =(1+\frac{k}{j}\ x)^j = 1 + \frac{1}{1}\ kx + \frac{1\cdot(j-1)}{1\cdot(2j)}\ k^2x^2 + \frac{1\cdot(j-1)(j-2)}{1\cdot(2j)(3j)}\ k^3x^3+. . . .$$

Now Euler points out that since j is infinitely large, $\frac{j-1}{j} = 1$, $\frac{j-2}{j} = 1$, $\frac{j-3}{j} = 1$, etc. Hence $\frac{j-1}{2j} = \frac{1}{2}$, $\frac{j-2}{3j} = \frac{1}{3}$, etc. This conclusion is an intuitive use of limits that is quite similar to the way Wallis drew his conclusions about characteristic ratio. Now (21) becomes:

$$(22) \quad a^x = 1 + \frac{k}{1}x + \frac{k^2}{1\cdot 2}x^2 + \frac{k^3}{1\cdot 2\cdot 3}x^3 + \frac{k^4}{1\cdot 2\cdot 3\cdot 4}x^4 + \ldots .$$

Letting x=1, (22) expresses the relationship between a (the base) and k (the slope).

$$(23) \quad a = 1 + \frac{k}{1} + \frac{k^2}{1\cdot 2} + \frac{k^3}{1\cdot 2\cdot 3} + \frac{k^4}{1\cdot 2\cdot 3\cdot 4} + \ldots .$$

Euler defines the number e as the base corresponding to the value of k=1. (22) now becomes:

$$(24) \quad e^x = 1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \ldots .$$

Using (23) he computes the value of the base e.

$$e = 2.71828182845904523536028 \ldots .$$

Looking back at the first part of (21) with k=1, we see the familiar statement that $e^x = \left(1 + \frac{x}{j}\right)^j$, or in modern terms $e^x = \lim_{j\to\infty}\left(1 + \frac{x}{j}\right)^j$.

Chapter VII of the *Introductio* also includes series that compute the inverses of (22) and (23). That is to say logarithmic series are demonstrated as well as a direct method for computing k given the value of the base a. Many examples are shown, with much discussion of their relative efficiency for actual calculation.

Unlike many mathematicians, Euler never tried to mask the possible pitfalls of his methods. In Chapter VII he gives an example of how seemingly correct algebra can lead to paradoxical results. The contradiction arises because an alternating series for a particular number actually diverges. Euler's advice is to proceed and faith will return. That is to say his approach to mathematics was empirical. Like Newton, he created equations by analogy and then tested them in various ways for to see if the results were consistent. Formal proofs of the conditions for the convergence of these binomial series were not given until nearly a century later by Gauss.

Throughout the *Introductio*, Euler made free use of complex numbers as well as the infinitely large and small quantities seen in the above example. He found that the series described above allowed him to extend the domain of e^x and ln(x) to the complex numbers. This ended a long controversy between Leibniz and Bernoulli concerning the appropriate definition of the ln(-1) (i.e. ln(-1)=iπ) (Cajori, 1913). In order to use binomial expansions to directly create series for sin(x) and cos(x), complex numbers are essential.

The derivation begins by factoring the identity $\sin^2(x)+\cos^2(x)=1$ into $[\sin(x)+i\cdot\cos(x)]\cdot[\sin(x)-i\cdot\cos(x)]=1$ (see Euler, 1988, chapter XIII). Euler then goes on to display the profound connections between trigonometric and exponential functions. Since the complex exponential maps vertical lines onto circles centered at zero, it becomes natural to write trig. functions as linear combinations of exponentials, i.e.

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \quad , \qquad \cos(x) = \frac{e^{ix} + e^{-ix}}{2} \quad .$$

Polar coordinates became very natural in this setting, and Euler makes extensive use of them in Book II of the *Introductio* .

An important point to consider here is that Wallis and Newton started with families of functions which through the use of extension, analogy, and interpolation gave rise to binomial series. Euler started with binomial series expansions, and by extension and analogy united exponential and trigonometric functions in one family. This beautiful circle of empirical investigations can be carried out at an elementary level and forms the grounded activity upon which first calculus and then differential equations were built.

Appendix 4: Negative Exponents and Ratios in Wallis

We have often found it interesting to examine some of the ideas in mathematics that did not gain general acceptance. The serious consideration of these alternative conceptions can enlighten our thinking and our teaching practice as we try to understand student conceptions. The following examination of Wallis' use of negative values within his theory of index and ratio is a good example.

Wallis interpreted negative numbers as exponents in the same way that we do. That is, he defined the index of $1/x$ as -1, and the index of $1/x^2$ as -2, and so on. He also extended this definition to fractions, for example $1/\sqrt{x}$ has an index of -1/2. He then claimed that the relationship between the index and the characteristic ratio is still valid for these negative indices. That is, that if k is the index then $1/(k+1)$ is the ratio of the area under the curve (shaded) to the rectangle (see figure 5a). In the case of a negative index this shaded area is unbounded. This did not deter Wallis from generalizing his claim.

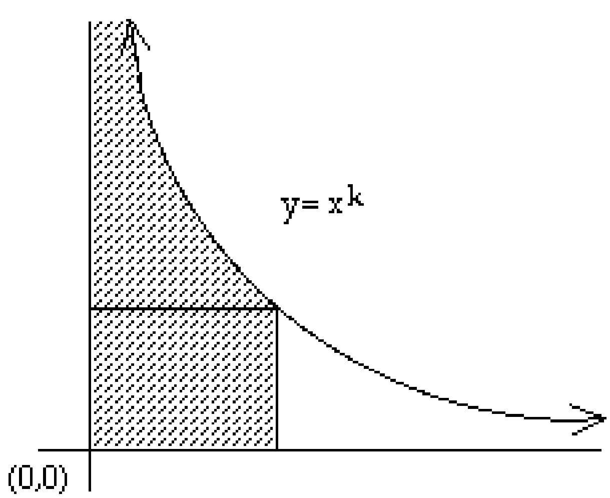

Fig. 5a

When k= -1/2, the characteristic ratio should be $1/(\frac{-1}{2} + 1) = 2$. This value is indeed correct, for the unbounded area under the curve $y = 1/\sqrt{x}$, does converge to twice the area of the rectangle. This is true no matter what right hand endpoint is chosen.

When k=-1, the characteristic ratio should be $1/(-1 +1) = 1/0 = \infty$ (Wallis introduced this symbol for infinity into mathematics). Wallis accepted this ratio as reasonable since the area under the curve $y = 1/x$, diverges. This can be seen from the divergence of the harmonic series $1+\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... = \infty$, which had been known since at least the fourteenth century (Boyer, 1968, Chap. XIV).

When k=-2, the characteristic ratio should be $1/(-2 + 1) = 1/-1$. Here Wallis' conception of ratio differs from our modern arithmetic of negative numbers. He did not believe that $1/-1 = -1$. Instead he stayed with his epistemology of multiple representations. Since the shaded area under the curve $y = 1/x^2$, is greater than the area under the curve $y =1/x$, he concluded that the ratio $1/-1$ is greater than infinity ("ratio plusquam infinita") (Nunn, 1909-1911, p. 355). He went on to conclude that $1/-2$ is even greater. This explains the plural in the title of his treatise *Arithmetica Infinitorum*.. The appropriate translation would be *The Arithmetic of Infinities* .

Most historians of mathematics quickly brush over this concept if they mention it at all. Those who mention it quickly site the comments of the French mathematician Varignon (1654 - 1722), who pointed out that if the minus sign is dropped in the ratio then we arrive at the correct ratio of the unshaded area under the curve to the area of the rectangle. This was an instance of the beginning of the idea that negative numbers could be viewed as complements or reversals of direction.

We, however, find it well worth pondering Wallis' original conception. In what ways does it make sense to consider the ratio of a positive to a negative number as greater than infinity ? In the area interpretation from figure 5a, we could view these different infinities as greater and greater rates of divergence. Such views are often taken in mathematics. The area under $y=1/x^3$ does diverge faster than the area under $y=1/x^2$.

Let's consider an even simpler situation. If I have $1, and you have 50¢, then we say that I have twice as much money as you. If I have $1, and you have 10¢ then we say that I have ten times as much money as you. If I have $1, and you have nothing, then we could say that I have infinitely more money than you. Many mathematicians would accept this statement. Now if I have $1, and you are in debt; shouldn't we say that the ratio of my money to yours is even greater than infinity ? This is a question that is worth pondering.

Appendix 5: Newton's Area Calculations

How did Newton know that he could create an area expression by summing up the area for each of the separate terms in a binomial expansion? He gave no reason at this point in the manuscript, but a reasonable reconstruction of thinking would most likely have been based on the area concepts of Calvalieri that are assumed in Wallis and in earlier manuscripts of Newton. Each individual power has its own characteristic

ratio but a sum of different powers has no such constant ratio, hence the area contributed by each term in an expansion must be considered as a fraction of a separate rectangle in order to use the results about characteristic ratio. Consider the total area under the curve
$y = ax^s + bx^t$, as the two separate pieces shown in figure 7a where the curve dividing the dark from the light area is $y=ax^s$.

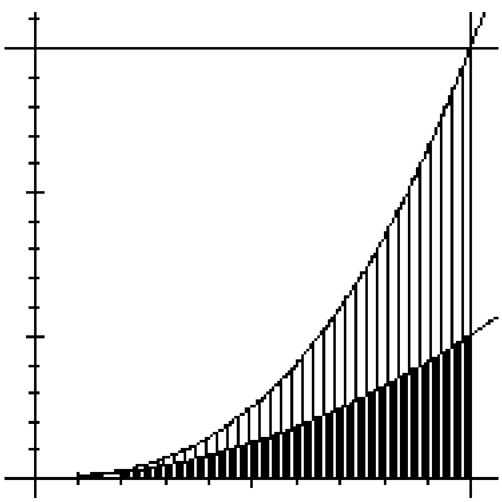

Fig. 7a

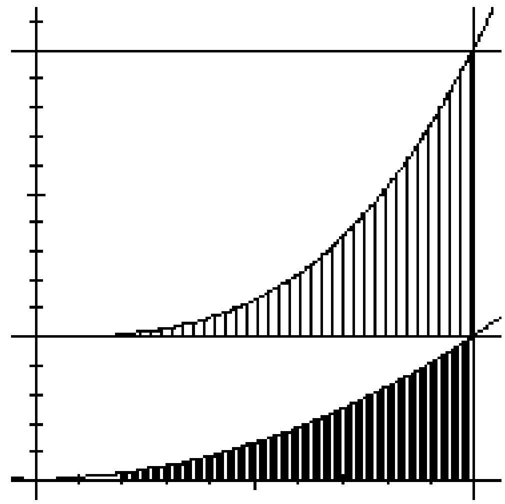

Fig. 8a

Leaving the darker area where it is we could now move each of the line segments that compose the lighter area up to the line y=k where k is the largest value of ax^s. (Think of moving the lighter area as if it were a deck of cards.) The lighter area will now fit inside a rectangle on top of the one that contains the darker area (see figure 8a). The area of the bottom rectangle is ax^{s+1}, and the area of the top rectangle is bx^{t+1}. From Wallis we know that the dark area is $\frac{1}{s+1}$ of the bottom rectangle and the lighter area is $\frac{1}{t+1}$ of the top rectangle, and hence the total area is $\frac{ax^{s+1}}{s+1} + \frac{bx^{t+1}}{t+1}$.

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