Functions of a Curve: Leibniz's Original Notion of Functions and Its Meaning for the Parabola

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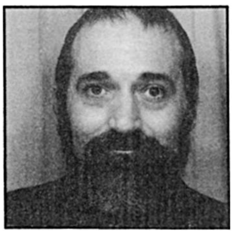

David Dennis received an M.S. in mathematical logic in 1977 under Anil Nerode, who sparked in him a great interest in the history of mathematics, which he especially enjoys teaching. He has held instructorships in mathematics (Ithaca College, Wells College, Queens College-CUNY), and for his recent Ph.D. in mathematics education (under Jere Confrey) his dissertation was on the history of curve drawing devices and their conceptual and educational significance. Cornell's math-education research group has given him new directions for his historical studies.

Jere Confrey, now an associate professor of mathematics education at Cornell University, founded the SummerMath program for women at Mount Holyoke College. How students view mathematical ideas intrigues her, since their views sometimes echo ideas long suppressed or forgotten since the rise of algebraic symbolism. She directs a research group that designs mathematical curriculum materials and software more inviting to all students. This group also conducts historical research to demystify the genesis of mathematical ideas.

When the notion of a function evolved in the mathematics of the late seventeenth century, the meaning of the term was quite different from our modern set theoretic definition, and also different from the algebraic notions of the nineteenth century. The main conceptual difference was that curves were thought of as having a primary existence apart from any analysis of their numeric or algebraic properties. Equations did not create curves, curves gave rise to equations. When Descartes published his *Geometry* [**10**] in 1638, he derived for the first time the algebraic equations of many curves, but never once did he create a curve by plotting points from an equation. Geometrical methods for drawing each curve were always given first, and then by analyzing the geometrical actions involved in the curve drawing apparatus he would arrive at an equation that related pairs of coordinates (not necessarily at right angles to each other) [**20**]. Descartes used equations to create a taxonomy of curves [**17**].

This tradition of seeing curves as the result of geometrical actions continued in the work of Roberval, Pascal, Newton, and Leibniz. Descartes used letters to represent various lengths but did not create any specific system of names. Leibniz, who introduced the term *function* into mathematics [**2**], considered six different functions associated with a curve, i.e., line segments or lengths that could be determined from each point on a curve relating it to a given line or axis. He gave them the names abscissa, ordinate, tangent, subtangent, normal, and subnormal. These six are shown in Figure 1 for the curve RP, relative to the axis AO. The line PO is perpendicular to AO. The line PT is tangent to the curve at P, and the line PN is perpendicular to PT.

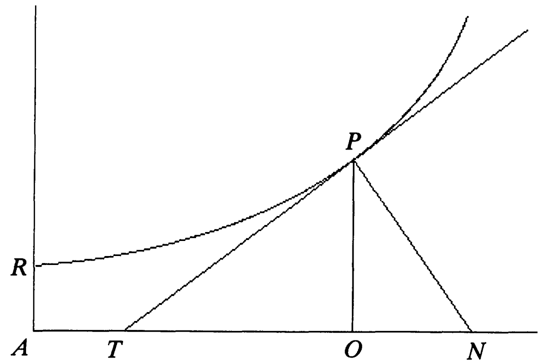

Figure 1
PO ordinate; *AO* abscissa; *PT* tangent; *OT* subtangent; *PN* normal; *ON* subnormal.

It is important to note here that the curve and an axis must exist before these six functions can be defined. In this definition, the abscissa and ordinate may at first seem to be a parametric representation of the curve, but this is not the case. No parameter, such as time or arc length, is involved. The setting is entirely geometric. From the geometric point P, the line segments (functions) are defined relative to the axis AO. *Abscissa* is Latin for "that which is cut off," i.e., a piece of the axis AO is cut off. By cutting off successive pieces of the axis, the curve gives us an ordered series of line segments PO as P moves along the curve. Hence the term *ordinate*.

It should also be noted here that all of these functions of a point P on a given curve are defined without reference to any particular unit of measurement. They are line segments. Leibniz, of course, like Descartes, wanted to introduce quantification and analyze the properties of curves algebraically, but since the definition of the functions is geometric he could postpone the choice of a unit until an appropriate one could be found for the curve at hand. The advantage of this will emerge in our discussion of the parabola.

Since angles *TPN*, *POT*, and *PON* are right angles, the triangles *TOP*, *PON*, and *TPN* are all similar. This configuration will be familiar to geometers as the construction of a geometric mean between ON and OT, the mean being OP.

Inspired by the work of Pascal, Leibniz saw a fourth triangle which was similar to the three mentioned above [**2**], [**5**], [**11**]. This was the *infinitesimal* or *characteristic* triangle (see Figure 2), used by Pascal to integrate the sine function [**21**]. Leibniz viewed a geometric curve as made up of infinitely small line segments which each had a particular direction. He perceived the utility of this concept in Pascal's work and it became one of the primary notions in his development of a system of notation for calculus. Although many modern mathematicians avoid this conception, it is still used as an important conceptual device by engineers. Figure 2 still appears in calculus books because it conveys an important meaning, especially to those who use calculus for the analysis of physical or mechanical actions. (With the invention, early in this century, of the calculus of differentials as linear functions on the tangent lines to the curve, Leibniz's fundamental insight was made rigorous without recourse to "infinitesimals" [**18**].)

Leibniz saw great significance in the triangles of Figure 1 because they were large and visible yet similar to the unseen characteristic triangle. This finding of large triangles that are similar to infinitesimal ones is a theme that runs through

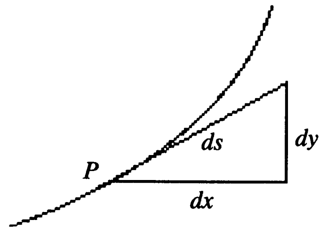

Figure 2

many of the most important works of Leibniz [**5**], [**8**], [**11**]. From Figures 1 and 2, the similarity relations tell us that

$$\frac{dy}{dx} = \frac{PO}{OT} = \frac{ON}{PO}.$$

Let us look at how this system works in the case of the parabola. We must first have a way to draw a parabola. Everything begins with the existence of a curve. Figure 3 shows a linkage that will draw parabolic curves. This figure comes from the work of Franz Van Schooten (1615–1660) [**23**, p. 359], whose extensive commentaries on Descartes' *Geometry* were widely read in the seventeenth century [**22**]. Because his works supplied many of the details Descartes omitted they were in fact more popular than the *Geometry* itself.

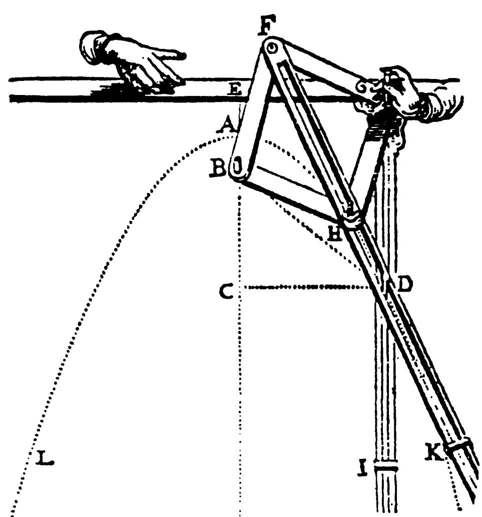

Figure 3

This apparatus constructs the parabola from the familiar focus/directrix definition. That is, the parabola is the set of points equidistant from a point and a line. The ruler GE is the directrix and the point B is the focus. Four equal-length links create a movable rhombus $BFGH$ which guarantees that FH will always be the perpendicular bisector of BG as G moves along the ruler. GI is a movable ruler that is always perpendicular to the directrix EG. The point D is the intersection of FH and GI as the point G moves along the directrix. Hence at all positions $BD = GD$, and hence D traces a parabola with focus B and directrix EG.

This construction can be simulated on a computer using the software *Geometer's Sketchpad* [**14**]. This software allows one to define a perpendicular bisector so the rhombus is unnecessary. One can either drag a point along the directrix or have the computer animate such a motion. Figure 4 was made using this software. The point F is the focus, and the point S is moving along the directrix. BP is the perpendicular bisector of FS, SP is always perpendicular to the directrix, and the intersection point P traces a parabola.

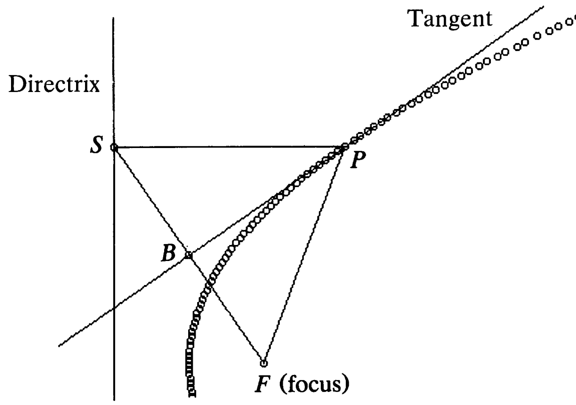

Figure 4

One consequence of this construction that is immediately apparent to the eye is that, at each point, BP is the tangent line to the curve at P. Curves can often be drawn by constructing a series of tangents to the curve, the curve being the "envelope" of its family of tangent lines. This construction is often done using strings or paper foldings [**13**], [**19**]. In order to fold a parabola as in Figure 4, let one edge of a sheet of paper be the directrix and mark any point as the focus. Make a series of folds each of which brings a point on the directrix onto the focus. These folds will then be the perpendicular bisectors of the segments between these pairs of points, hence tangent lines to the parabola.

Using the axis of symmetry of the parabola as our axis for abscissas and the vertex, A, as our starting point, we can investigate this curve using the six functions of Leibniz (Figure 5). Since the tangent line is part of the construction this can be

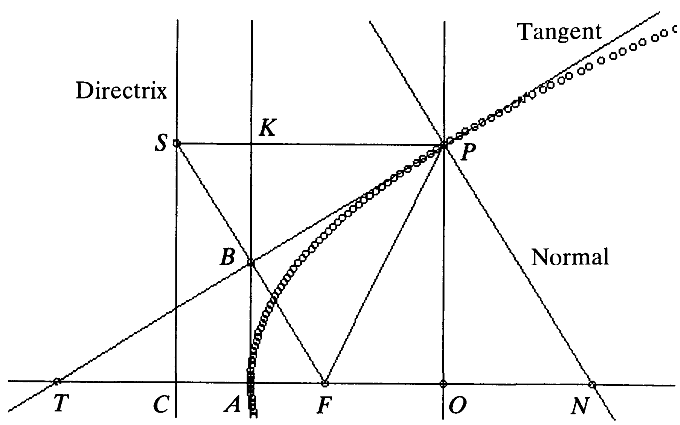

Figure 5

readily accomplished with *Geometer's Sketchpad*. Because it is impossible to convey the feel of this moving construction on paper, we strongly encourage the reader to experience it by dragging the point S up and down the directrix and observing how the "Leibniz configuration" changes.

What can be seen by watching the six functions in this dynamic setting? With the figure in motion and using color to highlight the six functions, two invariances become readily apparent. The first one most people notice is that the subnormal ON has constant length. The second is that the vertex A is always the midpoint of the subtangent OT, for points O and T can be seen to approach and recede from point A symmetrically. These two invariances can be easily deduced from the geometry of the construction, but of greater significance is that they can be visually experienced from the action of the construction. *Geometer's Sketchpad* allows for confirmation of one's visual experience by turning on meters that monitor these lengths empirically. Sure enough, ON has constant length, and the length of AT is always equal to the length of AO.

Postponing for a moment the geometrical proofs of these two statements, let us first look at what they tell us about the parabola. In the tradition of Descartes, we introduce variables after we have drawn the curve. Let $x = AO$, and let $y = OP$; i.e., x is the length of the abscissa and y is the length of the ordinate. Since triangles TOP and PON are similar, we have that $PO/OT = ON/PO$. Since A is the midpoint of OT, this becomes

$$\frac{y}{2x} = \frac{ON}{y}, \qquad \text{or} \qquad (2 \cdot ON) \cdot x = y^2.$$

Since ON is constant, this yields the equation of the parabola. The constant length $(2 \cdot ON)$ is known in geometry as the *latus rectum*; i.e., the rectangle formed by x and the latus rectum is always equal in area to the square on y. As we are free to choose our unit, we could choose $ON = \frac{1}{2}$. The equation then becomes $x = y^2$.

Using the similarity between the characteristic triangle and triangle TOP, we obtain

$$\frac{dx}{dy} = \frac{OT}{PO} = \frac{2x}{y} = 2y.$$

Hence both the equation and the derivative can be found from considering the invariant properties of Leibniz's configuration under the actions that constructed the curve.

The choice of $ON = \frac{1}{2}$ gave the equation and derivative of the parabola in their best known form, but this is perhaps a little artificial from the geometric standpoint. The subnormal ON is the primary invariant of this curve-drawing action and can be seen as the natural choice of a unit for this curve. As it turns out, the subnormal ON is always equal to the distance between the focus and the directrix of the parabola. Thus it is a natural unit. Using the subnormal as a unit, the equation of the parabola becomes $x = y^2/2$, i.e., the common integral form of the parabola as the accumulated area under the line $x = y$. It is in this form that the parabola most often appears in the table interpolations of John Wallis and Isaac Newton [**9**].

One way to prove that the subnormal is constant is to show that it always equals the distance between the focus and the directrix. Looking at Figure 5, we see that SF and PN are both perpendicular to BP, so triangles SCF and PON are congruent; hence $ON = CF$.

In order to prove that the vertex A is always the midpoint of the subtangent OT, one can establish that triangles TBA and PBK are congruent. They are clearly similar, but since B is the midpoint of SF it is also the midpoint of AK, so they are congruent. Hence $TA = KP = AO$.

Lastly, one might ask: How can we be sure that the line BP is always tangent to the parabola? That is to say, how can we be sure that each instance of the line BP intersects the parabola in only one point? Let $Q \neq P$ be a point on BP, and let R be the foot of the perpendicular from Q to the directrix CS. Since R is the closest point to Q on the directrix, $QR < QS$. Since BP is the perpendicular bisector of SF, $QS = QF$. Hence $QR < QF$ and Q cannot be on the parabola, being closer to the directrix than to the focus. One could also check the tangency of BP analytically by writing the equation of the parabola and the line BP using the same coordinate system and then solving the two equations simultaneously, arriving at a quadratic equation with one repeated root. This is the method that Descartes developed for finding tangents; i.e., tangency occurs when repeated roots appear in the simultaneous solutions.

These two invariant properties of the parabola were never mentioned (so far as we know) in the published work of Leibniz. The fact that the vertex is the midpoint of the subtangent was demonstrated by Appollonius [**1**]. The fact that the subnormal is constant is credited to L. Euler, who expanded and popularized the ideas of Leibniz [7]. They both appear in Book 2 of Euler's most famous textbook, the *Introduction to Analysis of the Infinite* [**12**]. This book, published in 1748, was the first modern precalculus textbook and, along with its sequels on differential and integral calculus, did much to standardize curriculum and notation. Nearly all of the topics in our modern precalculus books are contained in Euler's book, but what is missing from our modern treatments is the bold empirical spirit of Euler's investigations, as well as most of his more advanced geometry and infinite series. Euler says in the preface to his text that he presents many questions that can be more quickly resolved using calculus. He insists, however, that when students rush into calculus too rapidly they become confused, because they lack the experiential basis (both geometric and algebraic) upon which calculus is built.

The parabola example demonstrates how much can be found using only basic geometry combined with empirical investigation. By letting the configuration move, we create a situation where algebra evolves naturally from geometry. Too often in our schools we find our geometry curriculum static and isolated from other topics, especially algebra. Two-column geometry proofs provide a shadow of Euclid, but they cannot provide the dynamic experience that leads to an understanding of functions and calculus. An important philosophical prerequisite for understanding calculus is the belief that geometry and algebra are consistent with each other, and historically this belief did not come easily [**4**]. This belief is too often tacitly assumed in our classrooms. In order for students to comprehend and appreciate this they must first be allowed to experience doubt as to whether a geometric result will be confirmed by an arithmetic result [**8**]. With modern software, computers can now readily simulate moving geometry, and this experience can be very compelling. For some, an empirical experience based on mechanical devices or paper folding can be even more compelling.

For the reader who wishes to attempt this kind of analysis on other curves, we offer the following tantalizing tidbits. If the directrix in the above construction is a circle instead of a line, then one can draw both hyperbolas and ellipses with their tangents [**8**], [**23**]. Paper folding also works [**13**], [**19**]. In the case of the hyperbola, if a tangent line at a point P is extended until it intersects the asymptotes at points

A and B, then P will always be the midpoint of the segment AB. This little-known theorem is in Euler [**12**] but goes back to Apollonius [**1**]. As an empirical observation this can lead in many analytic directions. For example, the derivative of $y = 1/x$ can immediately be seen to be $-1/x^2$. Check it out! (Similar methods can be applied to draw planetary orbits; see the wonderful article by A. Lenard [**16**].)

Exercise. We have shown that parabolas have constant subnormals. What curves have constant subtangents? (Answer follows reference list.)

In order to have the kind of empirical experience that Lakatos [**15**] suggests is fundamental to mathematical discovery, people should be encouraged to design, build, and explore their own devices and computer simulations. Some experience with mechanical devices can greatly aid many students as they attempt to master the use of software like *Geometer's Sketchpad*. All algebraic curves, for example, can be drawn with linkages [**3**]; some are easily built and others are best simulated. The border between mathematics, simulation, and mechanical engineering can become quite fuzzy. In such a setting geometry and algebra complement, validate, and empower one another without forming a hierarchy.

After many years of working in mathematics education at all levels, we have come to believe that effective educational practice must involve people in a balanced dialogue between "grounded activity" and "systematic inquiry" [**6**]. This discussion of the parabola provides an excellent example of such a dialogue.

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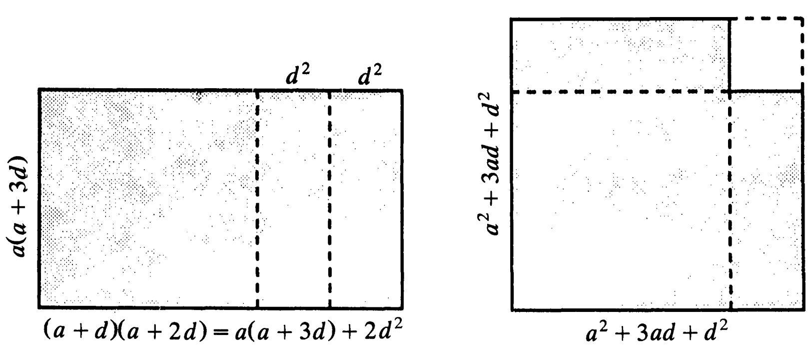
Answer to Exercise. Exponential curves $y = y_0 e^{kt}$. For a discussion of this question and many others like it, see [**8**].

The Product of Four (Positive) Numbers in Arithmetic Progression Is Always the Difference of Two Squares



$$a(a+d)(a+2d)(a+3d) = (a^2 + 3ad + d^2)^2 - (d^2)^2$$

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