The Creation of Continuous Exponents: A Study of the Methods and Epistemology of John Wallis

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I learned empirically that this came out this time, that it usually does come out; but does the proposition of mathematics say that? I learned empirically that this is the road I traveled. But is *that* the mathematical statement?-–What does it say, though? What relation has it to these empirical propositions? The mathematical proposition has the dignity of a rule.

Ludwig Wittgenstein (1967, p 47e)

Introduction:

Within a constructivist framework, we study student conceptions in order to develop approaches to concepts that make use of students' inventions and their creative use of resources. Our goal is to improve their understanding of mathematical ideas. However, we find that researchers who are traditionally educated in mathematics often fail to recognize or legitimate student methods, and need to broaden their understanding of possible routes towards the development of an idea. (We include ourselves in this description.) History has proven ideal for this, because it provides rich sources of alternative conceptualization and diverse routes to the development of an idea. Thus, we have found it to be a provocative and stimulating source of preparation for "close listening" to student mathematics (Confrey, 1993).

Moreover, such historical research invariably goes well beyond this original description. Our historical research guides us in building provocative tasks for our interviews. It leads us to make alternative proposals for curricular and instructional development. And it provides opportunities for teacher education: through the exploration of historical example, we can assist teachers in gaining depth in and perspective about their content knowledge.

But, perhaps most of all, our historical work leads us to reconceptualize our beliefs about the epistemology of mathematics. Our historical work affects our epistemological perspective, and our epistemological perspective influences the way that we engage in and interpret history. We would argue that the circularity in this argument is not a weakness but a necessity. Historical work serves to inform us about the present-- many of our current assumptions come to light through historical work. We recognize that there is no way that historical work can profess ultimate accuracy when recast by modern scholars. However, by use of original texts (when possible) and locating the work within the socio-cultural and historical context and by assuming a pluralistic history, we can attempt to understand it from the perspective of the originators. The methodological standards which are applied, are derived with respect to a set of epistemological assumptions which guide our overall research program.

The broad epistemological assumptions which underlie this historical investigation include the following:

1. A Lakatosian viewpoint that the development of mathematical ideas is characterized as a series of conjectures and refutations. Lakatos documented the limitations of a formalist methodology that hides the bold conjectures and masks the debates and refutations to present only the final product of a work. We take the view that mathematics is the process of proposing, developing, modifying and revising one's ideas, and that the proof, a normative assessment signaling the community's acceptance of an idea represents only a part of that process. Near the end of his life, Lakatos began to describe mathematics as quasi-empirical by which he referred to its progress via a set of heuristics. Whereas we understand and agree with Lakatos' concern that mathematics be viewed as more than a set of Euclidean axiomatics, we find his use of the term "quasi-empirical" somewhat ironic.

Much mathematics developed as a means to describe what scientists did and as a result, it is unclear where to draw the line between quasi-empirical and empirical. The modern mathematician R. Courant (1888 - 1972) wrote of C. F. Gauss that:

> "He (Gauss) was never aware of any contrast, not even of a slight line of demarcation, between pure theory and applications. His mind wandered from practical applications, undaunted by required compromise, to purest theoretical abstraction and back, inspiring and inspired at both ends. In light of Gauss' example, the chasm which was to open in a later period between pure and applied mathematics appears as a symbol of limited human capacity. For us today as we suffocate in specialization, the phenomenon of Gauss serves as an exhortation...it is critical for the future of our science that mathematicians adopt this course, both in research and in education." (Courant, 1984, p 132)

2. As an extension to a Lakatosian viewpoint, we view the history of mathematics as the coordination and contrast of multiple forms of representation. In history often one sees a particular form of representation as primary for the exploration, whereas another might form the basis of comparison for deciding if the outcome is correct. The confirming representation should be relatively independent from or contrasting to the primary exploratory representation. It must show contrast as well as coordination for the insight to be compelling. For example, Descartes most often generated his curves as loci of points tracing out certain movements in the plane. Algebra was used as the means for describing the curve, and as a result, he built his axis to be convenient for this description. The axes could be perpendicular or skewed depending on the situation. If a similarity argument was devised, a skewed axis was often built parallel to some line in the construction; if the Pythagorean theorem was used, a perpendicular axis was constructed (See Smith, Dennis and Confrey, 1992, or Dennis, 1995 for a further discussion of this). Thus, the modern Cartesian plane was not the starting point for Descartes' plane but was generated as a descriptive alternative representation. This sense is lost and distorted in our current curriculum where the Cartesian plane is treated predominantly as a means of displaying algebraic equations. This misrepresentation of history is not atypical.

In contrast, we propose to use the idea of an "**epistemology of multiple representations"** (Confrey, 1992 and Confrey & Smith, 1994). This approach leads us to examine the historical records in order to document such questions as what forms of representation were most influential for a given mathematician, what frameworks were used for confirmation, what representations were available as a part of "standard knowledge" for the time period, and how the mathematician moved among these representations to create and modify and extend mathematical pursuits. In reading this paper, one will see how the use of geometric insights into area and volume inform reasoning involving algebraic formula and how these are useful in confirming bold conjectures concerning patterns in a table. These table patterns later create an important transition that opens the door to admit infinite series into mathematical reasoning. Weaving these forms of representation together creates the framework that allows for progressive mathematical thought.

3. In contrast to what is explicitly developed by Lakatos, we believe that the socio-cultural period of the work exerts significant influence on the mathematical development. When possible within a scanty historical record, we will introduce such information.

Creating a reconstruction of the historical development of mathematical concepts is a difficult task for a number of reasons. These include a relative lack of access to original texts, the difficulty of interpreting original text as it appears notationally, the tendency to find secondary sources that distort the historical record by indiscriminately converting them to modern notation or form (see Unguru, 1976 for such a criticism) and a relative paucity of historical work in mathematics.[1]

We set our task in this paper to describe the development of rational exponents. We wished to understand this development at a deeper level than to simply assume that the development of rational exponents was a matter of extending a number pattern and its properties. What we found was that history reveals differences in the use of the ideas of powers, indexes, and exponents. In order for a generalized concept of continuous exponents to become accepted as legitimate mathematics it first had to be validated across several representations, those being table arithmetic, geometry, and algebra. The story of the creation of continuous exponents is linked strongly to the evolution of the notions of areas, limits of ratios, ratios with negative numbers, and continuous functions in general. Without this wider context only positive integer powers were accepted by a generation of mathematicians.

4. We did this historical research in light of the "splitting conjecture" (Confrey, 1988 and Confrey, 1994). According to this conjecture, Confrey postulated a different cognitive basis for splitting vs. counting. She suggested that basing multiplication in schools on repeated addition was neglecting the development of a parallel but related idea of equal sharing, of reproduction, magnification etc. Her splitting world includes the interrelated development of ratio, similarity, multiplication, division, multiplicative units, rates and exponential functions. Using this perspective, we conducted our historical investigation considering carefully how the use of geometry and ratio enlightened the development of mathematical thinking, seeking to avoid masking those distinctions in a generic algebraic description.

Historical Background:

The following is a work of historical investigation that focuses on specific mathematical moments where methods, concepts, and definitions underwent profound

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[1] For a good example of the kind of historical work that is required see Fowler (1987).

changes. There are two main goals. First to sketch the history of the development of a continuous concept of exponents, and second to examine carefully the epistemological setting in which these developments took place. Put more simply, what was it that convinced certain mathematicians that their concepts were viable. It is not intended to be a complete historical discussion, but rather a series of illuminating snapshots. The mathematical details of these moments are provided to the extent necessary for an understanding of the epistemology.

The main focus of this paper is on the mathematical methods of John Wallis (1606 - 1703), and in particular his influential work, the *Arithmetica Infinitorum*[2] (1972), first published in 1655. Although not the first person to suggest the use of fractional exponents, his work provided compelling reasons for their adoption. After reading Wallis, the young Isaac Newton (1642 - 1722) was inspired to derive his general binomial series (Whiteside, 1961). The binomial series was, in turn, the main tool used by Leonard Euler (1707 - 1783) to explore the world of continuous functions including natural based exponentials and logarithms (Euler, 1988). The work of all three of these men was carried out in an empirical setting without recourse to formal logical proofs. They checked and double-checked different representations against each other until they were convinced of the validity of their results. The formal proofs of the nineteenth century grew out of these results but often mask the methods.

Wallis openly advocated an empirical or heuristic approach to mathematical truth. He became convinced of the validity of his mathematics through a series of conjectures and confirmations. His main arguments depended upon a coordination of multiple representations. These included numerical tables, algebra, and geometry. For Wallis, a definition became reasonable when it emerged as a pattern in one representation but could also be confirmed through agreement with another. Just being a reasonable idea in one setting was never enough for Wallis. His primary investigations often took place in the setting of numerical sequences and tables. He then sought confirmation through algebra and geometry.

Wallis practiced induction, and by this word he did not mean formal mathematical induction, but informal or scientific induction. That is to say, he sought a pattern, checked a series of examples, and then assumed his rule was valid so long as he found "no ground of suspicion why it should fail" (Nunn 1909-1911, p.385). Formal proofs by mathematical induction were being carried out by his contemporaries Fermat (1601 - 1665), and Pascal (1623 - 1662). Fermat criticized the methods of Wallis as suggestive but incomplete. Wallis responded, saying that he was trying to develop a theory of knowledge that was far superior to the logical analysis of known results. He claimed that Fermat "doth wholly mistake that treatise (i.e. the *Arithmetica Infinitorum*) which was not so much to show a method of Demonstrating things already known as to show a way of Investigation or finding of things yet unknown" (quoted in Nunn 1909-1911, p.385). Wallis felt that the ancient mathematicians were in possession of such a method but that it was "studiously concealed" (Nunn 1909-1911, p.385) and covered over with logical analysis.

Wallis was concerned with action rather than logical justification. In this he was part of a general philosophical movement away from the Greek based, neoplatonic thought of the Renaissance. In the seventeenth century Roman texts became far more popular than Greek ones. The works of Seneca and other Roman stoic philosophers

[2] In 2004 J.A. Stedall published an English translation of *Arithmetica Infinitorum.*

were revived. Roman thought is generally much more practical and empirical than Greek thought. Clear and direct language is highly valued.

Wallis was an expert in many languages. Among others he mastered Latin, Greek, Hebrew, and probably Arabic. He first came to prominence during the civil wars in England. He aided the Parliamentary Party by rapidly deciphering coded Royalist messages that had been captured in 1642 (Scott ,1981, p.6). This service got him a post at Oxford during the rule of Oliver Cromwell. An instinct for patterns and language marks all of his work. As you read the work of Wallis consider how a cryptographer works. He need not provide logical proof that his interpretation is correct, rather, when an interpretation makes sense his work is finished.

In order to understand the portions of Wallis that we will present it is necessary to know the polynomial formulas for the summation of the integer powers of the first n integers. These formulas just appear in Wallis' *Arithmetica Infinitorum* (1972) with no mention of where he got them or how to derive them. Mathematicians in the seventeenth century rarely provided any references, and historians have speculated on Wallis' various possible sources for these formulas. They appear in a number of seventeenth century French sources including works by Fermat, and Pascal (Boyer, 1943). Another interesting possible source for Wallis may have been his reading of Arabic texts. Wallis did read Arabic, and these formulas appeared in the work of ibn-al-Haitham (965 - 1039), known in the West as Alhazen. Wallis definitely had access to some of the mathematical work of Alhazen, but historians have not been able to directly verify exactly which texts he read.[3]

Since the focus of this paper is to use historical material for educational purposes, we will begin with a derivation of these summation formulas that is based on a text by Alhazen (Baron, 1969; Edwards, 1979). This method for deriving these formulas fits beautifully with the concepts that appear in the *Arithmetica Infinitorum*, although only circumstantial evidence exists as to what sources Wallis actually read.[4] When discussing the work of Wallis, we will stay very close to the form and notation of the original works. The work of Alhazen, however, will not be discussed in its original form. The Arab mathematicians did not develop symbolic algebra, thus all of the formulas of Alhazen would have been written out in words. For brevity, the derivations of Alhazen are given in modern algebraic form.

It must be noted here that even our presentation of the derivations of Alhazen is somewhat controversial. The geometric figures that we will present in the next section do not actually appear in the Arabic texts of Alhazen. Some historians (Katz, 1993) have interpreted this portion of a work by Alhazen as purely arithmetical, while others (Baron, 1969) have interpreted this work geometrically, as we will do. In an age of handwritten manuscripts, many geometrical works (including Euclid) often lacked figures; containing instead instructions for the construction of figures. One simply could not trust scribes to accurately reproduce figures. These summation formulas are derived in the beginning of a work by Alhazen that is purely geometrical (concerned with the volumes of paraboloids), and in general the work of Alhazen is much more geometrical than other Arabic mathematicians of his time. For these reasons, it seems to us that Baron's geometrical interpretation of these formulas is reasonable.

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[3] Recently Victor Katz has investigated this question in detail.

[4] Victor Katz, for example, is inclined to believe that Wallis got these formulas from the work of Faulhaber, but then the question arises as to what sources were used by Faulhaber.

Putting aside these questions of historical authenticity, the main reason that we have chosen to begin with these derivations springs from our educational focus. If students are to enter into the table interpolations of John Wallis and really understand their significance, they must be able to understand and accept the summation formulas that he took for granted. Of all of the ways that we know of to derive these formulas, this opening interpretation of a work of Alhazen seems best suited to our educational purposes, since it produces these algebraic formulas by linking them to a geometrical representation.

Alhazen's Summation Formulas and Powers Higher than Three:

Alhazen was physicist as well as a mathematician. His work was grounded in the Greek geometrical tradition, but he also sought after practical empirical results in optics and astronomy. Greek mathematics (with the exception of Diophantus and Heron) does not mention any powers higher than three because they could not be directly interpreted in geometry, i.e. as lengths, areas, or volumes. Alhazen wanted to calculate new results concerning areas and volumes which involved the summation of powers higher than three. For example he computed the volume generated by rotating a parabola about a line perpendicular to its axis of symmetry (see Figure 1). For a modern student this would involve integrating a fourth power polynomial. Alhazen stated that the volume is 8/15 of the volume of the circumscribed cylinder. In the tradition of Eudoxus, he set up upper and lower sums of cylindrical slices, and then let the slices get finer and finer. Since the radius of each cylindrical slice follows a square function, the areas follow a fourth power. The sum of the areas of these slices involves summing fourth powers (Edwards, 1979, p.85).

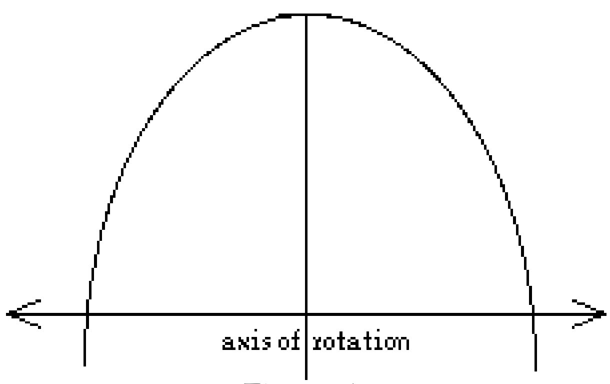

Figure 1

Alhazen derived formulas for the sums of higher powers by coordinating a geometrical interpretation with a numerical representation. His use of areas to represent third and fourth powers broke with the strict geometrical interpretations in Greek mathematics. His method can be used to find a formula for the sum of the powers of the first n integers for any positive integer power. His extension of the concept of powers made sense arithmetically, but was validated through his derivations of new geometric results.

Alhazen derives his formulas by first laying out a sequence of rectangles whose areas represent the terms of the sum (Edwards, 1979, p.84). A rectangle of area a^k is formed using sides of length a^{k-1} and a. He then fills in the rectangle with a series of interlocking strips (see Figures 1, 2 and 3, which are to scale). Alhazen then sets the product of the dimensions of the rectangle equal to the sum of its rectangular parts. Each of the formulas can then be derived from the previous ones. The strips on top, however, involve a double summation in all but the first derivation.

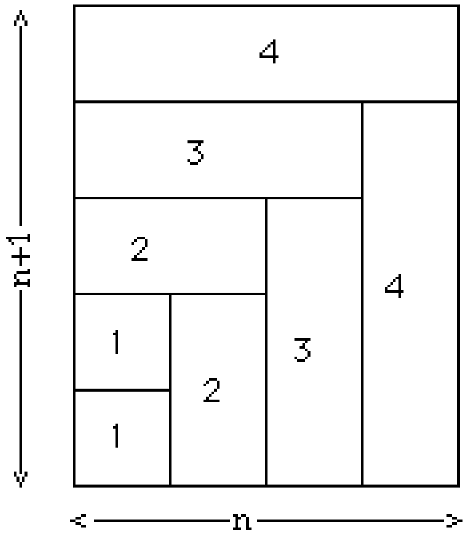

Figure 2

To obtain a formula for sum of the first n integers, see Figure 2.

$$n\,(n+1) = \sum_{i=1}^{n} i + \sum_{i=1}^{n} i$$

(1) $$\frac{1}{2}\,n\,(n+1) = \frac{1}{2}\,n^2 + \frac{1}{2}\,n = \sum_{i=1}^{n} i = 1+2+3+ \ldots\ldots +n$$

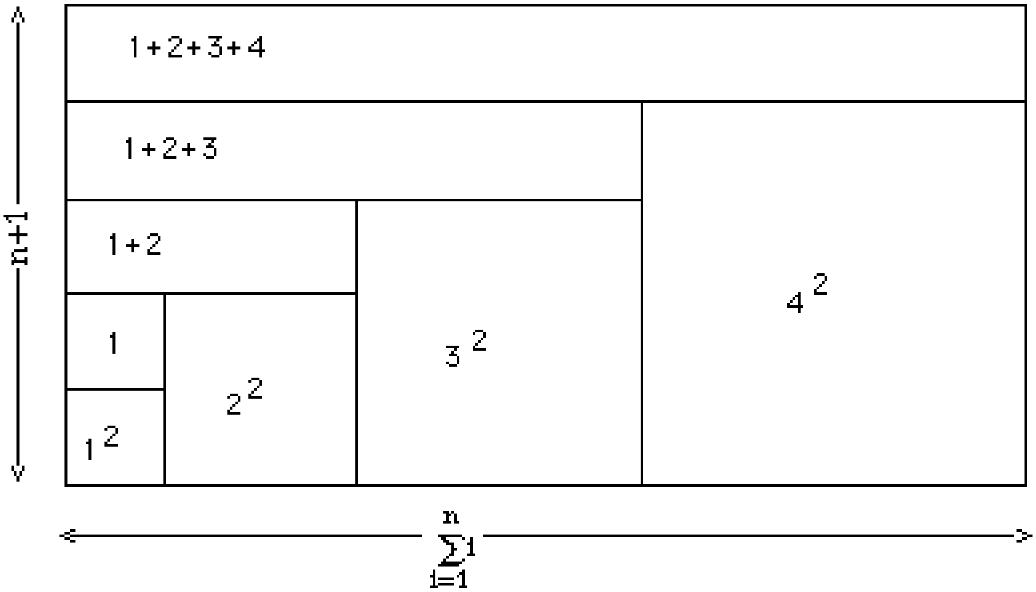

Figure 3

To obtain a formula for the sum of the squares of the first n integers, see Figure 3.

$$\left(\sum_{i=1}^{n} i\right)(n+1) = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n}\left\{\sum_{k=1}^{i} k\right\}$$

$$\left(\frac{1}{2}n^2 + \frac{1}{2}n\right)(n+1) = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n}\left\{\frac{1}{2}i^2 + \frac{1}{2}i\right\}$$

$$\frac{1}{2}n^3 + n^2 + \frac{1}{2}n = \sum_{i=1}^{n} i^2 + \frac{1}{2}\sum_{i=1}^{n} i^2 + \frac{1}{2}\left(\frac{1}{2}n^2 + \frac{1}{2}n\right)$$

$$\frac{1}{2}n^3 + \frac{3}{4}n^2 + \frac{1}{4}n = \frac{3}{2}\sum_{i=1}^{n} i^2$$

$$(2) \qquad \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+ \ldots\ldots\ldots\ldots+n^2$$

Note that (1) was used twice in obtaining (2).

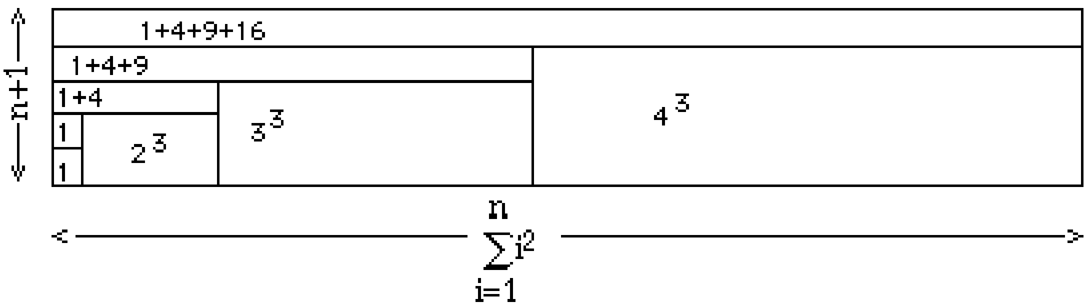

Figure 4

To obtain a formula for the sum of the cubes of the first n integers, see Figure 4.

$$\left(\sum_{i=1}^{n} i^2\right)(n+1) = \sum_{i=1}^{n} i^3 + \sum_{i=1}^{n}\left\{\sum_{k=1}^{i} k^2\right\}$$

$$\left(\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n\right)(n+1) = \sum_{i=1}^{n} i^3 + \sum_{i=1}^{n}\left\{\frac{1}{3}i^3+\frac{1}{2}i^2+\frac{1}{6}i\right\}$$

$$\frac{1}{3}n^4+\frac{5}{6}n^3+\frac{4}{6}n^2+\frac{1}{6}n = \sum_{i=1}^{n} i^3 + \frac{1}{3}\sum_{i=1}^{n} i^3 + \frac{1}{2}\left(\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n\right) + \frac{1}{6}\left(\frac{1}{2}n^2+\frac{1}{2}n\right)$$

$$\frac{1}{3}n^4 + \frac{4}{6}n^3 + \frac{4}{12}n^2 = \frac{4}{3}\sum_{i=1}^{n} i^3$$

$$(3) \qquad \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 = \sum_{i=1}^{n} i^3 = 1^3+2^3+3^3+ \ldots\ldots\ldots\ldots +n^3$$

Note that here (2) was used twice and (1) was used once.

A formula for the sum of the first n fourth powers can be derived by continuing this method. First lay out a series of rectangles whose horizontal sides are the cubes and whose vertical sides are once again successive integers. Filling in the strips and proceeding as before will yield the desired formula, although the algebra becomes more tedious. All three of the previous formulas must be used. This derivation is left as an exercise for the reader. This method can be extended to yield a formula for the sum of the first n powers for any integer power. It is a general recursion scheme for these formulas, but at each stage not just one but many of the previous formulas must be used. For future reference here are Alhazen's Formulas:

(1) $1+2+3+ ....+n= \frac{1}{2} n^2 + \frac{1}{2} n$

(2) $1^2+2^2+3^2+ ....+n^2 = \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n$

(3) $1^3+2^3+3^3+ ....+n^3 = \frac{1}{4} n^4 + \frac{1}{2} n^3 + \frac{1}{4} n^2$

(4) $1^4+2^4+3^4+ ....+n^4 = \frac{1}{5} n^5 + \frac{1}{2} n^4 + \frac{1}{3} n^3 - \frac{1}{30} n$

In order to calculate the volume of the rotated parabola in Figure 1, the actual summation needed is $\sum_{i=1}^{n}(n^2-i^2)^2 = \frac{8}{15} n^5 - \frac{1}{2} n^4 - \frac{1}{30} n$, which can be derived from (2) and (4) by substituting and collecting terms. The formula inside the summation is the square of the radius of each slice (Edwards, 1979, p.84).

It should be stressed here that results about areas and volumes were not given as formulas but always as ratios. This is true about nearly all such mathematical results until the end of the seventeenth century. For example the area of a triangle is one half of the area of the parallelogram that contains it. The volume of a pyramid is one third of the box that contains it. The area of a piece of a parabola is two thirds of the rectangle that contains it. The area of the circle is $\pi/4$ of the square that contains it. These examples were all known by the second century BC. Alhazen's ratio of 8/15 is a continuation of that tradition, but he had to coordinate geometry and arithmetic in a new way to find it. In the work of John Wallis we will see an elaborate consideration of area ratios used to justify his definition of fractional and negative exponents. He made extensive use of these summations, and in the final validation of his ideas, Alhazen's ratio of 8/15 appeared in one of his tables.

The *Arithmetica Infinitorum* of John Wallis:

The juxtaposition of arithmetic and geometric sequences goes back at least as far as Aristotle. The idea that one could insert values into such a table by using arithmetic means on one side and geometric means on the other is also ancient. The concept of fractional exponents is suggested in various ways in the works of Oresme (14th cent.), Girard and Stevin (16th cent.) (Boyer 1968, chapters XIV, and XV). The building of tables of logarithms by Napier and others in the early 17th century implies the possibility of calculating such powers although this was not how Napier himself interpreted his work. It is important to note that these earlier discussions took place within the world of numerical tables and there was no compelling coordination with geometry. Since geometry was then the dominant form of mathematics, these early works never led to the development or acceptance of fractional exponents.

The Geometry (1952), first published in 1638, of René Descartes was the first published treatise to use positive integer exponents written as superscripts. Descartes saw exponents as an index for repeated multiplication. That is to say he wrote x^3 in place of xxx. Wallis adopted this use of an index and tried to extend it, and test its validity across multiple representations.

Wallis took from Fermat the idea of using an equation to generate a curve, which was in contrast to Descartes' work which always began with a geometrical construction. Descartes always constructed a curve geometrically first, and then analyzed it by finding its equation (Smith, Dennis, and Confrey, 1992). Fermat independently developed coordinate geometry, but his approach was more algebraic (Mahoney, 1973). He started with equations and then looked at the curves they generate. This approach is taken by Wallis. His treatment of conic sections, for example, is very similar to that of Fermat (Boyer, 1956).

Wallis proposed the notion of fractional indices (exponents) and showed how this notion could be validated in both algebraic and geometric settings (1972). Consideration of areas under curves provided Wallis with a alternative representation with which to validate his proposed arithmetical definition of fractional indices. Wallis' definitions had a lasting impact on mathematics because he demonstrated their viability across multiple representations. Finding the areas under curves was an old problem going back to Archimedes. Beginning in the fourteenth century with Oresme the problem of finding the area under a curve became important because curves were used to represent the magnitudes of velocity over time. The area under the curve then represented the total change in position. This context was laid out in the 14th century by Oresme in his *Latitude of Forms* (Calinger, 1982, p. 224). In the early seventeenth century Galileo used this concept extensively.

Geometry was considered the primary representation of mathematics in the seventeenth century. This is evident in the work of Barrow and others of that time (Boyer, 1968). Arithmetic and algebra were considered, at best, to be forms of shorthand language for the discussion of geometric truth. Some scholars even doubted whether arithmetic and geometry could ever be made consistent. Wallis reverses this order and considers arithmetic as his primary representation (Cajori, 1929). In order to validate his arithmetical results he had to show that they yielded the accepted geometric conclusions. To do this he first derived a series of arithmetical ratios and then deduced from them many of the known ratios of areas and volumes.

The *Arithmetica Infinitorum* (1972) contains a detailed investigation of the behavior of sequences and ratios of sequences from which a variety of geometric results are then concluded. We shall look at one of the most important examples. Consider the ratio of the sum of a sequence (of a fixed power) to a series of constant terms all equal to the highest value appearing in the sum. Wallis considered ratios of the form:

$$(5) \qquad \frac{0^k+1^k+2^k+....+n^k}{n^k+n^k+n^k+....+n^k}$$

For each fixed integer value of k, Wallis investigated the behavior of these ratios as n increases. His investigations are empirical in character. For example when k=1, he calculates:

$$\frac{0+1+2}{2+2+2}=\frac{1}{2} \qquad \frac{0+1+2+3}{3+3+3+3}=\frac{1}{2} \qquad \frac{0+1+2+3+4}{4+4+4+4+4}=\frac{1}{2} \qquad \text{etc.}$$

As r increases this ratio stays fixed at 1/2. This can be seen from the well known summation formula (1) in a factored form. The numerator is $\frac{n(n+1)}{2}$, while the denominator is n(n+1). Wallis calls 1/2 the characteristic ratio of the index k=1.

When k=2, Wallis computed the following ratios:

$$\frac{0^2+1^2}{1^2+1^2} = \frac{1}{3} + \frac{1}{6}$$

$$\frac{0^2+1^2+2^2}{2^2+2^2+2^2} = \frac{1}{3} + \frac{1}{12}$$

$$\frac{0^2+1^2+2^2+3^2}{3^2+3^2+3^2+3^2} = \frac{1}{3} + \frac{1}{18}$$

$$\frac{0^2+1^2+2^2+3^2+4^2}{4^2+4^2+4^2+4^2+4^2} = \frac{1}{3} + \frac{1}{24}$$

$$\frac{0^2+1^2+2^2+3^2+4^2+5^2}{5^2+5^2+5^2+5^2+5^2+5^2} = \frac{1}{3} + \frac{1}{30}$$

Wallis claimed that the right hand side is always equals $\frac{1}{3} + \frac{1}{6n}$. To see this he applied another summation formula (2) in a factored form. The numerator is always equal to $\frac{1}{6}$ n(n+1)(2n+1), while the denominator is equal to $n^2(n+1)$. As n increases this ratio approaches 1/3, so Wallis then defined the characteristic ratio of the index k=2 as equal to 1/3. In a similar fashion Wallis computed the characteristic ratios of k=3 as 1/4, and k=4 as 1/5. He then made the general claim that the characteristic ratio of the index k is $\frac{1}{k+1}$ for all positive integers.

In all of his examples Wallis started with the sequence {0,1,2,3,4,…} and then raised each term to the index under consideration. Wallis asserted, however, that his values for the characteristic ratios are valid for any arithmetic sequence starting with zero. Changing the difference between the terms would only introduce a constant multiple into all of the terms in both the numerator and the denominator and hence the ratios would remain unchanged. For example, the sum of the squares of any arithmetic sequence (starting at zero) divided by an equal number of terms all equal to the highest term (e.g. $\frac{0^2+2^2+4^2+6^2}{6^2+6^2+6^2+6^2}$); would still approach 1/3 as more terms are taken.

Wallis then went on to show that these characteristic ratios yielded most of the familiar ratios of area and volume known from geometry. It is here that he showed that his arithmetic was consistent with the accepted truths of geometry. His basic assumptions about the nature of area and volume were taken from Cavalieri's *Geometria Indivisibilibus Continuorum* (1635). He assumed that an area is a sum of an infinite number of parallel line segments[5], and that a volume is a sum of an infinite number of parallel areas. Wallis first considered the area under the curve $y= x^k$ (See Figure 5). He wanted to compute the ratio of the shaded area to the area of the rectangle which encloses it.

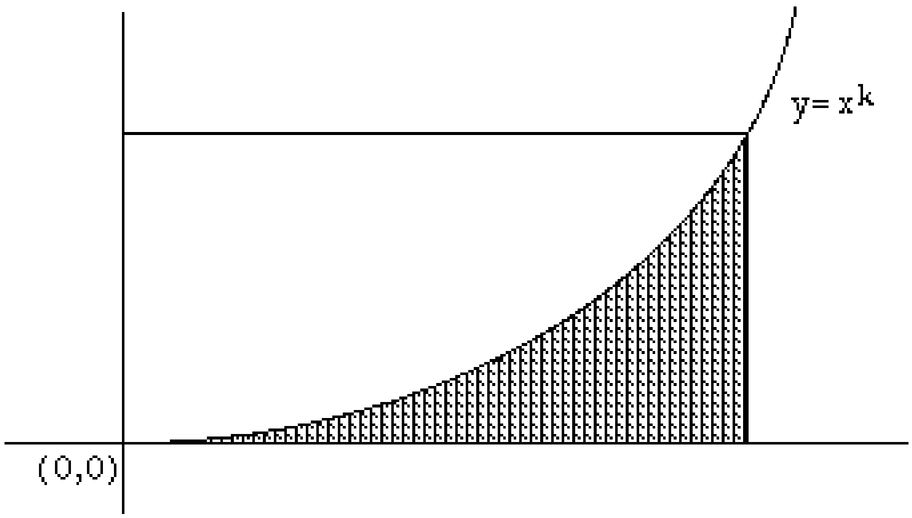

Figure 5

Wallis claimed that this geometric problem is an example of the characteristic ratio of the sequence with index k. The terms in the numerator are the lengths of the line segments that make up the shaded area while the terms in the denominator are the lengths of the line segments that make up the rectangle (hence constant). Since his ratios are valid for any arithmetic sequence, he imagined the increment or scale as very small while the number of the terms is very large. Hence, for example, the area under a parabola is 1/3 of the area of the rectangle.[6] This geometric ratio is exactly 1/3 because the area is made up of an infinite number of line segments. It should be noted here that this characteristic ratio of 1/3 holds for all parabolas, not just $y=x^2$. For example if we look at the ratio calculation for $y=5x^2$, both the area under the curve and the area of the rectangle are multiplied by 5, and hence the characteristic ratio of 1/3 remains the same. Characteristic ratio depends only on the exponent and not on the coefficient. That is to say, characteristic ratio is not linear.

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[5] This conception of a curve as a series of line segments erected along a line gave rise to the terms 'abscissa' and 'ordinate'. Abscissa is Latin for 'that which is cut off' and contains the same root as our word 'scissors'. Ordinates are the ordered series of line segments which are being erected from that which was cut off (abscissa). These terms were coined by Leibniz.

[6] The skeptical reader could fix the interval from 0 to M, and then let n+1 be the number of equal size sub-intervals. Set up the Riemann lower sum divided by the area of the rectangle. Canceling common factors you will arrive at (5). Riemann sums, however, are based on sums of areas of rectangles and this was not Wallis' conception of area.

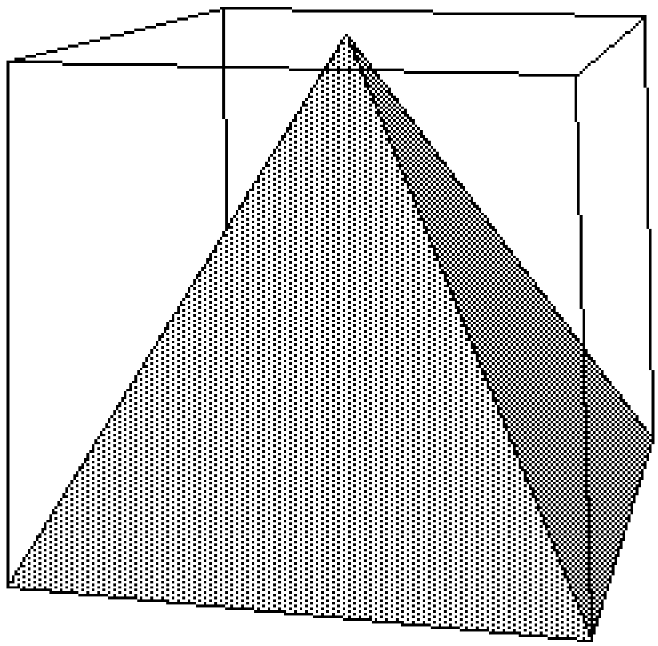

Figure 6

This characteristic ratio of x^2 also shows that the volume of a pyramid is 1/3 of the box that surrounds it (see Figure 6). The pyramid is the sum of a series of squares whose sides are increasing arithmetically. The box is a series of squares whose sides are constant and always equal to the largest square. Hence Wallis sees this as another example of his computation of the characteristic ratio for the index k=2. The same is true for the ratio of the cone to its surrounding cylinder. Here all of the terms both top and bottom are multiplied by π.

These geometric results were not new. Fermat, Roberval, Cavalieri and Pascal had all previously made this claim that when k is a positive integer; the area under the curve $y=x^k$ had a ratio of 1/k+1 to the rectangle that encloses it (Edwards, 1979, Chap. 4). Pascal had given a formal induction proof of this result. Wallis, however, goes on to assert that if we define the index of $\sqrt{x}$ as $\frac{1}{2}$, the claim remains true. Since the area under the curve $y=\sqrt{x}$ is the complement of the area under $y=x^2$ (i.e. the unshaded area in figure 5), it must have a characteristic ratio of $\frac{2}{3} = \frac{1}{\frac{1}{2}+1}$. The same can be seen for $y=\sqrt[3]{x}$, whose characteristic ratio must be $\frac{3}{4} = \frac{1}{\frac{1}{3}+1}$.

It was this coordination of two separate representations that gave Wallis the confidence to claim that the appropriate index of $y=\sqrt[q]{x^p}$ must be $\frac{p}{q}$, and that its characteristic ratio must be $\dfrac{1}{\frac{p}{q}+1}$. Wallis went on to assert that this claim remained true even when the index is irrational. He looked at one such example; that of an index equal to the $\sqrt{3}$.

In many cases, Wallis had no way to directly verify the characteristic ratio of an index, for example: $y=\sqrt[3]{x^2}$. It is here that he invokes his principle of "interpolation." He coined this term from the Latin root "to polish." He claimed that whenever one can discern a pattern of any kind in a sequence of examples one has the right to apply that pattern to any intermediate values if possible. That is to say that one should always attempt to polish in between. Nunn (1909 -1911) calls this his principle of continuity, and claims that this is a major step towards the development of a theory of continuous functions. Its influence on Newton was profound.

Wallis did proceed boldly with his principle of interpolation, but he always sought some way to double-check his patterned conjectures through an interpretation outside of his original representation. It is this confirmation through and across representations that made his interpolations so compelling. He tried to construct a continuous theory which would connect all the little islands of accepted truth. With this in mind let us look at a subsequent section of the *Arithmetica Infinitorum*, which contains his most famous interpolation.

How can we determine the characteristic ratio of the circle? This is the question that motivated Wallis to study a particular family of curves from which he could interpolate the value for the circle. He wrote the equation of the circle of radius r, as $y=\sqrt{r^2-x^2}$, and considered it in the first quadrant. He wanted to determine the ratio of its area to the r by r square that contains it. Of course he knew that this ratio is $\pi/4$, from various geometric constructions going back to Archimedes, but he wanted to test his theory of index, characteristic ratio and interpolation by arriving at this result in a new way. This would provide a confirmation of the theory through coordination of representations.

As it stands the equation of the circle does not yield to the methods he has developed thus far. Every student of calculus confronts this when he/she finds that the general power rule will not integrate the circle. Wallis searched for a family of equations in which he could embed the circle equation. He considered the family of curves defined by the equations $y = \left(\sqrt[q]{r} - \sqrt[q]{x}\right)^p$. This family is binomial, symmetrical and includes the circle as a fractional case (p=q=1/2).[7] Figure 7 shows the graphs of Wallis' equations in the unit square (r=1) for p=1/2, q=0, 1/2, 1, 3/2,...,5 and for q=1/2, p=0, 1/2, 1, 3/2,...,5. The line y=x has been added to display the symmetry.

[7] The symmetry of this family of curves can be seen by rewriting the equations in the form
$y^{1/p} + x^{1/q} = r^{1/q}$. Reversing x and y is the same as reversing p and q. This form also displays their relation to the equation of the circle in the form $y^2+x^2=r^2$.

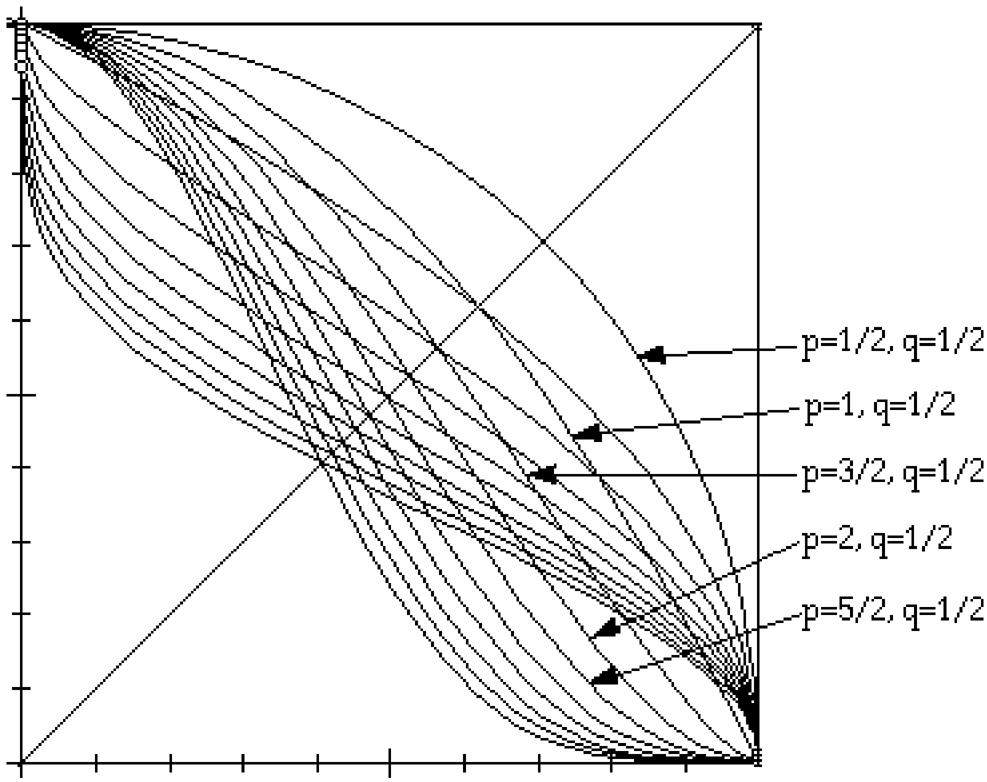

Figure 7

If p and q are both integers, he knew that by expanding the binomial $(\sqrt[q]{r} - \sqrt[q]{x})^p$ to the p^{th} power and using his rule for characteristic ratios he could determine the ratio for these curves. Figure 8 shows this as the ratio of the shaded area to the rectangle which encloses it. Note that all of these curves pass through (r,0), and that the height of the rectangle is $\sqrt[q]{r^p}$.

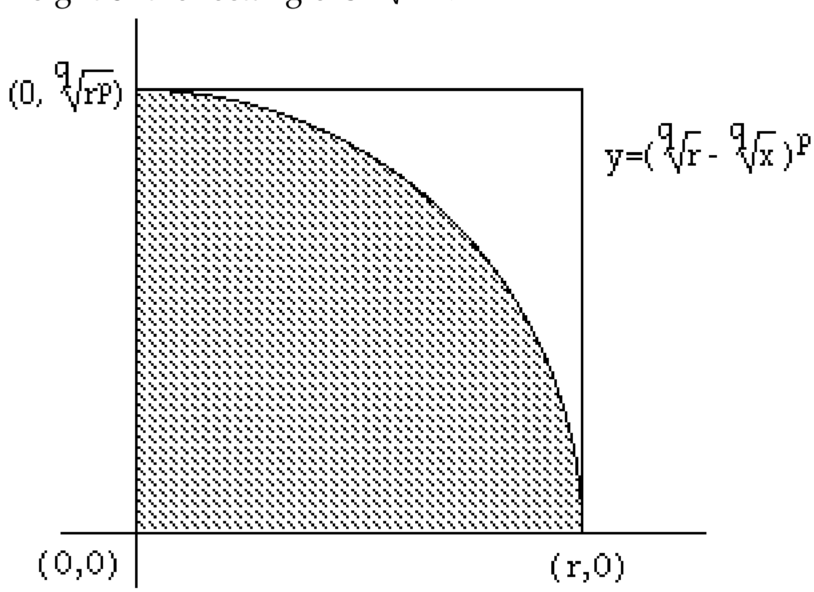

Figure 8

The circle can be seen as the case when p =1/2, q = 1/2. Note here that Wallis is interpreting $\sqrt[\frac{1}{2}]{x} = x^2$. He also allowed p or q to be zero. He gave the interpretation $x^0=1$ based on his characteristic ratio rule. Since $y=x^0$ must have a characteristic ratio of one, it must be a horizontal line. Since one to any power is one this horizontal line must be at height one.

Wallis then calculated the characteristic ratio when p, and q are both integers. For example, if p=q=2, then: $y=(\sqrt{r} - \sqrt{x})^2 = r - 2\sqrt{r}\sqrt{x} + x$, and so must have a characteristic ratio of $1 - 2 \cdot \frac{2}{3} + \frac{1}{2} = \frac{1}{6}$. Several things must be noted about this calculation. First, Wallis used the sequence of x values: {0,1,2,3,...,r }, and then let r increase. He assumed that the calculation was valid for any arithmetic sequence. Hence his characteristic ratio is independent of the value of r.[8] This he confirmed empirically by computing a vast array of examples. Second, when looking at the middle term, it may appear at first glance that since the characteristic ratio of $\sqrt{x}$ is 2/3, that the -2 comes along for the ride in a linear fashion. But as we said before, characteristic ratios are not linear. The characteristic ratio of $2\sqrt{x}$ is still 2/3. However, in this case the calculation is valid since the maximum value of the entire curve is determined by the constant term r. Since all of the values in the denominator are r, the coefficient of the middle term does not factor out of the denominator, and hence does not cancel out.

After computing the characteristic ratios when p and q are integers, Wallis noted that they were all unit fractions and that the denominators were the figurate or binomial numbers. These numbers had been known since ancient times and had recently been discussed by Pascal and others in the seventeenth century. Wallis then inverted these ratios so that they became integers and made a table of them. Table one records the ratio of the rectangle to the shaded area (see Figure 8) for each of the curves $y = (\sqrt[q]{r} - \sqrt[q]{x})^p$.

Table 1

q\p	0	1/2	1	3/2	2	5/2	3	7/2	4
0	1		1		1		1		1
1/2									
1	1		2		3		4		5
3/2									
2	1		3		6		10		15
5/2									
3	1		4		10		20		35
7/2									
4	1		5		15		35		70

[8] One could regard 'r' as the scaling factor for the system of coordinates which are being imposed upon a preexisting curve.

At this point Wallis temporarily abandoned both the geometric and algebraic representations and began to work solely in the table representation. The question then became, how does one interpolate the missing values in this table? [9]

Wallis proceeded as follows. He first worked on the rows with integer values of q. When q=0, we see the constant value of one, so fill it in with ones. When q=1, we see an arithmetic progression, so fill it in with arithmetic means. In the row q=2 we have the triangular numbers which are the sums of the integers in the row q=1. Hence he could use the formula for the sum of consecutive integers, $\frac{s^2+s}{2}$, where $s=p+1$. This formula appeared in margin of his table. Putting the intermediate values into this formula allows us to complete the row q=2. For example letting s=3/2 in this formula yields 15/8, which becomes the entry where p=1/2, and q= 2.

The numbers in the row q=3 are the pyramidal numbers each of which is the sum of the integers in the row q=2. Hence the appropriate formula is found by summing the formula from the row q=2. Applying Alhazen's first two formulas and then collecting terms Wallis obtained $\frac{1}{2}\sum_{1}^{s} i^2+i = \frac{s^3+3s^2+2s}{6}$, which appears in the margin of his row q=3. Letting s=3/2, for example, the formula yields 105/48, which becomes the table entry where p=1/2 and q=3 (again s=p+1).

In a similar fashion Wallis summed the previous cubic formula to obtain a formula for the row q=4. Using the first three of Alhazen's formulas and then collecting terms we obtain: $\frac{s^4+6s^3+11s^2+6s}{24}$.

This procedure completes rows where q is an integer by applying the formulas to the intermediate values. Since the table is symmetrical this also allows us to fill in the corresponding columns when p is an integer. Wallis then created the following table, with his formulas for interpolation written in the margin.[10]

[9] At this point the reader may wish to try his/her own hand at completing this table. If these ideas were to be used with students this could generate a fascinating discussion.
[10] To view the original table of Wallis, see Wallis, 1972, p. 458; or Struik, 1969, p. 252.

Table 2

q \ p	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	
0	**1**	1	**1**	1	**1**	1	**1**	1	**1**	1
$\frac{1}{2}$	1		$\frac{3}{2}$		$\frac{15}{8}$		$\frac{105}{48}$		$\frac{945}{384}$	
1	**1**	$\frac{3}{2}$	**2**	$\frac{5}{2}$	**3**	$\frac{7}{2}$	**4**	$\frac{9}{2}$	**5**	s
$\frac{3}{2}$	1		$\frac{5}{2}$		$\frac{35}{8}$		$\frac{315}{48}$		$\frac{3465}{384}$	
2	**1**	$\frac{15}{8}$	**3**	$\frac{35}{8}$	**6**	$\frac{63}{8}$	**10**	$\frac{99}{8}$	**15**	$\frac{s^2+s}{2}$
$\frac{5}{2}$	1		$\frac{7}{2}$		$\frac{63}{8}$		$\frac{693}{48}$		$\frac{9009}{384}$	
3	**1**	$\frac{105}{48}$	**4**	$\frac{315}{48}$	**10**	$\frac{693}{48}$	**20**	$\frac{1287}{48}$	**35**	$\frac{s^3+3s^2+2s}{6}$
$\frac{7}{2}$	1		$\frac{9}{2}$		$\frac{99}{8}$		$\frac{1287}{48}$		$\frac{19305}{384}$	
4	**1**	$\frac{945}{384}$	**5**	$\frac{3465}{384}$	**15**	$\frac{9009}{384}$	**35**	$\frac{19305}{384}$	**70**	$\frac{s^4+6s^3+11s^2+6s}{24}$

The more familiar rule for the formation of a binomial table tells us that each entry should be the sum of two others, one of which appears here two spaces up, and the other two spaces to the left. One can now check that the new interpolated values also conform to this rule of formation. Note here that the 15/8 occurs in the place p=2, q=1/2. The area under this curve involves exactly the same summation that occurs in the calculation of the volume of Alhazen's paraboloid, and the interpolation is consistent with Alhazen's result.

With these entries now in place, Wallis turned his attention to the row q=1/2. Each of the entries that now appear there is calculated by using each of the successive interpolation formulas that appear in the margins. Each of these formulas has a higher algebraic degree. What pattern exists in the formation of these numbers which will allow us to interpolate between them to find the missing entries? [11] Remember that the first missing entry is the q=p=1/2 (i.e. the ratio of the square to the area of the quarter circle), and so if our table manipulations are to be validated back in the geometric representation this value should come out to be $4/\pi$.

$q=\frac{1}{2}$	1		$\frac{3}{2}$		$\frac{15}{8}$		$\frac{105}{48}$		$\frac{945}{384}$

At first Wallis tried to fill in this row with arithmetic averages. The average of 1 and 3/2 is 5/4. But if $4/\pi$ is equal to 5/4, then $\pi = 3.2$, which is not quite correct. If geometric averaging is attempted, the value for π comes out even larger (≈ 3.266).

Wallis now observed that each of the numerators in these fractions is the product of consecutive odd integers, while each of the denominators is the product of consecutive even integers. That is to say $\frac{15}{8} = \frac{3\cdot 5}{2\cdot 4}$, $\frac{105}{48} = \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}$, and $\frac{945}{384} = \frac{3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8}$.

Hence to move two entries to the right in this row one multiplies by $\frac{n}{n-1}$. For the entries that appear so far, n is always odd. So Wallis assumed that to get from one missing entry to the next one he should still multiply by $\frac{n}{n-1}$, but this time n would have to be the intermediate even number. Denoting the first missing entry by Ω,[12] then the next missing entry should be $\frac{4\Omega}{3}$, and the one after that should be $\frac{4\cdot 6}{3\cdot 5}\,\Omega = \frac{8}{5}\,\Omega$, and the one after that should be $\frac{4\cdot 6\cdot 8}{3\cdot 5\cdot 7}\,\Omega = \frac{64}{35}\,\Omega$ and so forth. The q=1/2 row now becomes:

$q=\frac{1}{2}$	1	Ω	$\frac{3}{2}$	$\frac{4}{3}\Omega$	$\frac{15}{8}$	$\frac{8}{5}\Omega$	$\frac{105}{48}$	$\frac{64}{35}\Omega$	$\frac{945}{384}$

The column p=1/2 can now be filled in by symmetry. The row q=3/2 has a similar pattern (i.e. products of consecutive odd over consecutive even numbers) but there the entries two spaces to the right are always multiplied by $\frac{n}{n-3}$. One can also double check, as Wallis did, that this law of formation agrees with usual law for the formation of binomials, i.e. each entry is the sum the entries two up, and two to the left. The full table now becomes:

[11] Once again, we would encourage the reader to stop and try his/her own hand at filling in the missing entries.

[12] Wallis used a small square to denote this missing entry.

Table 3

q \ p	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	
0	**1**	1	**1**	1	**1**	1	**1**	1	**1**	1
$\frac{1}{2}$	1	Ω	$\frac{3}{2}$	$\frac{4}{3}\Omega$	$\frac{15}{8}$	$\frac{8}{5}\Omega$	$\frac{105}{48}$	$\frac{64}{35}\Omega$	$\frac{945}{384}$	
1	**1**	$\frac{3}{2}$	**2**	$\frac{5}{2}$	**3**	$\frac{7}{2}$	**4**	$\frac{9}{2}$	**5**	s
$\frac{3}{2}$	1	$\frac{4}{3}\Omega$	$\frac{5}{2}$	$\frac{8}{3}\Omega$	$\frac{35}{8}$	$\frac{64}{15}\Omega$	$\frac{315}{48}$	$\frac{128}{21}\Omega$	$\frac{3465}{384}$	
2	**1**	$\frac{15}{8}$	**3**	$\frac{35}{8}$	**6**	$\frac{63}{8}$	**10**	$\frac{99}{8}$	**15**	$\frac{s^2+s}{2}$
$\frac{5}{2}$	1	$\frac{8}{5}\Omega$	$\frac{7}{2}$	$\frac{64}{15}\Omega$	$\frac{63}{8}$	$\frac{128}{15}\Omega$	$\frac{693}{48}$	$\frac{512}{35}\Omega$	$\frac{9009}{384}$	
3	**1**	$\frac{105}{48}$	**4**	$\frac{315}{48}$	**10**	$\frac{693}{48}$	**20**	$\frac{1287}{48}$	**35**	$\frac{s^3+3s^2+2s}{6}$
$\frac{7}{2}$	1	$\frac{64}{35}\Omega$	$\frac{9}{2}$	$\frac{128}{21}\Omega$	$\frac{99}{8}$	$\frac{512}{35}\Omega$	$\frac{1287}{48}$	$\frac{1024}{35}\Omega$	$\frac{19305}{384}$	
4	**1**	$\frac{945}{384}$	**5**	$\frac{3465}{384}$	**15**	$\frac{9009}{384}$	**35**	$\frac{19305}{384}$	**70**	$\frac{s^4+6s^3+11s^2+6s}{24}$

The table is now complete except for the determination of the value of Ω. One might be tempted to think that we are done, since we know from geometric considerations that Ω must equal to 4/π. Wallis already knew from Archimedes one way to evaluate π by inscribing polygons in the circle. This ancient method of calculating π leads, algebraically, to a series of nested square roots, one for each doubling of the number of sides in the polygon. Using this geometric result at this point would, however, violate the whole program of Wallis. He had to find a way to calculate Ω using his principle of interpolation so that he could check his value against the one known from geometry. It is only in this way that he created a critical experiment capable of testing and validating his method of interpolation.[13]

So returning once again to the row q=1/2 where moving two spaces to the right from the n^{th} entry multiplies that entry by $\frac{n}{n-1}$, Wallis noted that as n increases the fraction $\frac{n}{n-1}$ gets closer and closer to 1. Hence the number two spaces to the right must change very little as we go further out the sequence. This is true of the calculated fractions as well as the multiples of Ω. Wallis argued that since the whole sequence is monotonically increasing, that consecutive terms must also be getting close to one another as we proceed. Hence as we build these terms we should have that:

$$\Omega \; \frac{4\cdot 6\cdot 8\cdot 10....}{3\cdot 5\cdot 7\cdot 9....} \approx \frac{3\cdot 5\cdot 7\cdot 9....}{2\cdot 4\cdot 6\cdot 8....}$$

or $$\Omega \approx \frac{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9...}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10...}.$$

Since Ω should be equal to 4/π, Wallis' interpolations are justified provided that:

$$\pi \approx 2\cdot\frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8....}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9....}$$

If one calculates this infinite product it does indeed converge to π. That is it agrees with the results of Archimedes and others who had previously calculated π within a geometric representation. It is this final coordination of multiple representations that confirmed for Wallis the viability of his principle of interpolation. This was an entirely new way to calculate π, and it became famous. It is often mentioned in modern analysis books and its connections to other topics are discussed, but rarely does anyone mention its epistemological significance. This calculation is a critical empirical experiment which confirms the consistency of geometry and arithmetic. This, as Wallis would say, has been "studiously concealed" by logical analysis. Wallis wanted his work to convey the usefulness of empirical methods of investigation within mathematics. To just see this result as a new way to calculate π is to "wholly mistake the design of the treatise" (Nunn, 1909-1911, p.385).

The empirical methods of Wallis led the young Isaac Newton to his first profound mathematical creation; the expansion of functions in binomial series (Whiteside, 1961). The interested reader should look at Newton's annotations to Wallis contained in his early notebooks to see how far this method of interpolation can lead (Newton, 1967). Wallis' method of interpolation became for Newton the basis of his notion of continuity. Newton generalized the methods of Wallis, but for several generations, the epistemology used by Wallis remained the dominant force in mathematics. The elaborate investigations of Euler in the eighteenth century, which

[13] The philosopher Thomas Hobbes criticized the work of Wallis as a "scab of symbols." Hobbes saw no reason why the results of algebra should be consistent with those of geometry. The response of Wallis to these comments was a great embarrassment to the aging philosopher (Hobbes). See Cajori, 1929.

included series expansions over complex numbers and the solution of many differential equations, nonetheless remained rooted in an empirical epistemology of multiple representations (Euler, 1988).

Conclusions:

The methods of investigation outlined in this paper have been largely purged from our mathematics curriculum. These mathematical results are now presented to students in a formal logical setting that came about in the nineteenth and twentieth centuries. They are presented as a done deed; a finished object. Almost no sense of the activity of investigation remains. Both for the mathematical specialist and the non-specialist this type of presentation is dominant in our mathematics classrooms.

The practice of conjecture by analogy, and the use of informal induction combined with coordination of multiple representations would greatly invigorate our teaching practices. For the non-specialist many practical methods of mathematical reasoning are entirely concealed. So many results of mathematics could be presented much earlier and in a simpler setting if empirical methods were encouraged. Even for the student who chooses to specialize in mathematics most of the process of invention is never highlighted, though much of the work of a professional mathematician is carried out in this way.

The availability of calculators, and computers make it possible for many people to engage in empirical speculations. Indeed many of the most interesting area of modern research in mathematics are being carried out as computer experiments. The fractal geometry developed by Mandelbrot and others is a good example. Astounding new pictures have been constructed that involve little formal analysis. These empirical methods are now being applied profoundly in biology. Recently some of the most original mathematical research has come from outside of math departments.

In summary we would say that it is the mathematical action rather than the result which most deeply conveys meaning. It is the construction of a mathematical setting where these actions can be checked across different representations that produces confidence in the viability of a method. It is the understanding of method that empowers students. Logical analysis can be very beautiful and satisfying in its own way, but it is like a spider spinning its webs in the castle of mathematics. The danger is that after a while one begins to believe that the webs hold up the castle.[14] Students exposed only to traditional, overly formal curriculum are especially prone to this danger.

Appendix - Negative Exponents and Ratios:

We have often found it interesting to examine some of the ideas in mathematics that did not gain general acceptance. The serious consideration of these alternative conceptions can enlighten our thinking and our teaching practice as we try to understand student conceptions. The following examination of Wallis' use of negative values within his theory of index and ratio is a good example.

Wallis interprets negative numbers as exponents in the same way that we do. That is, he defines the index of $1/x$ as -1, and the index of $1/x^2$ as -2, and so on. He also extends this definition to fractions, for example $1/\sqrt{x}$ has an index of -1/2. He

[14] David Dennis heard this aphorism many years ago at a seminar, but can not remember who said it.

then claims that the relationship between the index and the characteristic ratio is still valid for these negative indices. That is, if k is the index then 1/(k+1) is the ratio of the area under the curve (shaded) to the rectangle (see Figure 9). In the case of a negative index this shaded area is unbounded. This does not deter Wallis from generalizing his claim.

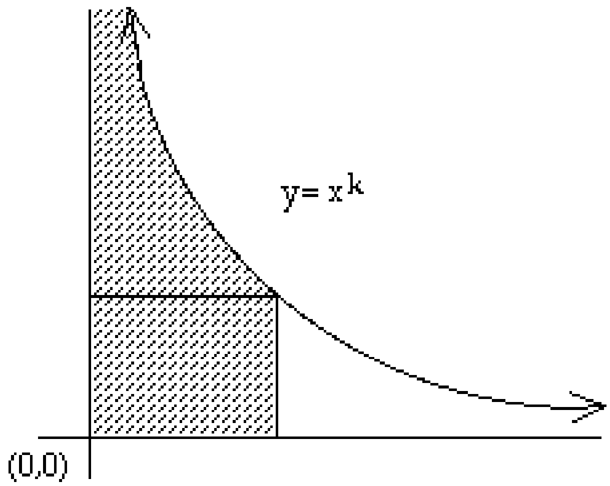

Figure 9

When k= -1/2, the characteristic ratio should be $1/(\frac{-1}{2}+1) = 2$. This value is indeed correct, for the unbounded area under the curve $y = 1/\sqrt{x}$, does converge to twice the area of the rectangle. This is true no matter what right hand endpoint is chosen.

When k=-1, the characteristic ratio should be $1/(-1 +1) = 1/0 = \infty$ (Wallis introduced this symbol for infinity into mathematics). Wallis accepted this ratio as reasonable since the area under the curve $y = 1/x$, diverges. This can be seen from the divergence of the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+ .... = \infty$, which had been known since at least the fourteenth century (Boyer, 1968, Chap. XIV).

When k=-2, the characteristic ratio should be $1/(-2 + 1) = 1/-1$. Here Wallis' conception of ratio differs from our modern arithmetic of negative numbers. He does not believe that $1/-1 = -1$. Instead he stays with his epistemology of multiple representations. Since the shaded area under the curve $y = 1/x^2$, is greater than the area under the curve $y =1/x$, he concludes that the ratio 1/-1 is greater than infinity ("ratio plusquam infinita") (Nunn, 1909-1911, p. 355). He goes on to conclude that 1/-2 is even greater. This explains the plural in the title of his treatise *Arithmetica Infinitorum*.. The appropriate translation would be *The Arithmetic of Infinities*.

Most historians of mathematics quickly brush over this concept if they mention it at all. Those who mention it quickly site the comments of the French mathematician Varignon (1654 - 1722), who pointed out that if the minus sign is dropped in the ratio then we arrive at the correct ratio of the unshaded area under the curve to the area of the rectangle. This was an instance of the beginning of the idea that negative numbers could be viewed as complements or reversals of direction.

We, however, find it well worth pondering Wallis' original conception. In what ways does it make sense to consider the ratio of a positive to a negative number as greater than infinity? In the area interpretation from Figure 9, we could view these different infinities as greater and greater rates of divergence. Such views are often taken in mathematics. The area under $y=1/x^3$ does diverge faster than the area under $y=1/x^2$.

Let's consider an even simpler situation. If I have \$1, and you have 50¢, then we say that I have twice as much money as you. If I have \$1, and you have 10¢ then we say that I have ten times as much money as you. If I have \$1, and you have nothing, then we could say that I have infinitely more money than you. Many mathematicians would accept this statement. Now if I have \$1, and you are in debt; shouldn't we say that the ratio of my money to yours is even greater than infinity? This seems to us to be a question that is worth pondering.

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